Spring 2020

Rationality Questions and the Derived Category

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by

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Bachelor of Science
Shippensburg University of Pennsylvania 2015

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2020

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Acknowledgments

First and foremost, I would like to thank my advisor Dr. Matthew Ballard, who has for the entirety of graduate school encouraged me to challenge myself, and provided unwavering support, guidance, and patience. Without him, this document would not exist. I must also thank Dr. Alexander Duncan and Dr. Patrick McFaddin, who I have been fortunate enough to work alongside and learn from. I am also indebted to all of my committee members and their involvement in my graduate career.

I would not be where I am without the encouragement and support of three incredibly talented teachers at North Hagerstown High School: Mrs. Cheryl Cross, Ms. Brandy Merchant, and Mr. David Warrenfeltz. Thank you all for setting three different, albeit all equally as wholesome, examples of what it means to be a teacher.

I must also express gratitude to Dr. Lenny Jones, who has been a friend and a mathematical mentor to me for nearly a decade. I cannot possibly thank you enough for all of the time and effort (and wine) that you have invested into my mathematics career. Thank you for believing in me for so long.

I am very thankful for the support of my family, friends, and for the wonderful community of algebraic/arithmetic geometers that I have had the pleasure of interacting with during my time as a student. I would also like to thank my best friend, Candace Bethea, who has made graduate school the best time of my life. Thank you for teaching me so much about mathematics, and for always being by my side.

Abstract

This document is roughly divided into four chapters. The first outlines basic preliminary material, definitions, and foundational theorems required throughout the text. The second chapter, which is joint work with Dr. Matthew Ballard, gives an example of a family of Fano arithmetic toric varieties in which the derived category is able to detect the existence of k-rational points. More succinctly, we show that if X is a generalized del Pezzo variety defined over a field k, then X contains a k-rational point (and is in fact k-rational, that is, birational to $\mathbb{P}^n_k$) if and only if $\mathsf{D}^{\mathsf{b}}(X)$ admits a full étale exceptional collection.

In the third chapter, which is joint work with Dr. Matthew Ballard, Dr. Alexander Duncan, and Dr. Patrick McFaddin, we describe, using techniques from Galois cohomology, a new invariant of reductive algebraic groups that captures precisely when this strategy will fail. Our main result characterizes this invariant in terms of coflasque resolutions of linear algebraic groups introduced by Colliot-Thélène. We determine whether or not this invariant is trivial for many fields. For number fields, we show it agrees with the Tate-Shafarevich group of the linear algebraic group, up to behavior at real places. In addition, we completely describe the cohomological invariants of a reductive algebraic group of degree 2 with values in a special torus, which generalizes a result of Blinstein and Merkurjev.

In the final chapter, which is joint work with Dr. Matthew Ballard, Dr. Alexander Duncan, and Dr. Patrick McFaddin, we develop tools to understand the effect that twisting by a torsor has on the derived category. Applying these to the setting of arithmetic toric varieties reveals a surprising dichotomy in behavior split along the

fault line of retract rationality. As a consequence of the general theory, we give a negative answer to a question of Bernardara and Bolognesi relating a categorical notion of dimension to rationality. Moreover, we show that a smooth projective toric variety over a field k possessing a full k-exceptional collection is automatically k-rational.

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Chapter 1

Preliminaries

1.1 Introduction

Determining the existence of solutions to a system of equations is a fundamental problem in mathematics that has historically provided mathematicians with a wealth of surprisingly difficult phenomena to ponder. At the heart of this is the following question: when does a system of polynomials with integer coefficients have integer solutions?

A budding mathematician may learn in an elementary number theory class that the linear equation $ax + by = c$ for $a, b, c \in \mathbb{Z}$ has a solution if and only if $\gcd(a, b)$ divides c (given that $a, b \neq 0$). For quadratics, the *Hasse-Minkowski Theorem* tells us that integral solutions exist if and only if there exist solutions over every completion of $\mathbb{Q}$; that is, there exist real solutions and solutions in $\mathbb{Q}_p$ for every prime p. This is an example of what is now known as the *Hasse principle*, the ability to stitch together local solutions, e.g. solutions over $\mathbb{Q}_p$, to get a global solution, e.g. a solution over $\mathbb{Q}$. Although this is a desirable property to have, it does not always hold. For example, consider $x^4 - 17 = 2y^2$, which has solutions in $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes p, but no solutions over $\mathbb{Q}$.

Algebraic geometers are also interested in solutions sets to systems of polynomials, although they are frequently studied under the guise of *algebraic varieties*. The *sheaf cohomology* of a variety is in many ways a geometric analogue of the Hasse principal: it measures the failure of the ability to glue together local solutions to a geometric problem to obtain a global solution over the entire variety. If we are looking for obstructions to the existence of rational points, it is therefore natural to turn to a well-studied repository of the cohomological information attached to a variety X, $\mathsf{D}^{\mathsf{b}}(X)$- its derived category of coherent sheaves. This leads one to wonder the following: over a non-algebraically closed field k, what sort of arithmetic properties does the variety X enjoy, and can these be detected by the derived category of coherent sheaves on X? In broad strokes, the goals of this dissertation are grounded in the desire to investigate arithmetic properties revealed via the derived category of a variety.

1.1.1 Notation and Conventions.

Throughout this text, F will denote a field and its separable closure will be written as $\overline{F}$. We let Γ be the absolute Galois group of F equipped with its profinite topology. Unless otherwise specified, we assume a *variety* X is a geometrically integral separated scheme of finite type over F. For a scheme X defined over F and a field extension L of F, we write:

$$X_L := X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(L)$$

to be the *base extension* of X, which is an L-scheme. If L is the separable closure of F, we write instead $\overline{X}$.

For a triangulated category T, we will write $\mathrm{Ext}^n_{\mathsf{T}}(A, B) = \mathrm{Hom}_{\mathsf{T}}(A, B[n])$. For objects A, B of $\mathsf{D}^{\mathsf{b}}(X)$, we write

$$\mathrm{End}_X(A) = \mathrm{End}_{\mathsf{D}^{\mathsf{b}}(X)}(A), \text{ and}$$

$$\mathrm{Ext}^n_X(A, B) = \mathrm{Ext}^n_{\mathsf{D}^{\mathsf{b}}(X)}(A, B).$$

1.2 Toric Varieties

As mentioned in the introduction, algebraic geometers are broadly interested in the vanishing set of a collection of polynomial equations, which is called an algebraic variety. A small, useful family of algebraic varieties are toric varieties; which contain a torus as a dense open subset. A unique and incredibly robust feature of toric varieties is that they offer a way to translate geometric questions into purely combinatorial information. This is akin to carving handholds in a sheer stone wall so that one can climb it. Climbing the wall is still difficult, but it is something that can be done if enough effort is made. Because of this, mathematicians are able to explicitly compute important geometric invariants associated to a particular toric variety using purely combinatorial data. Generally speaking, these invariants are very difficult, if not impossible to calculate in most other situations. In this section, we will introduce the foundational definitions and theorems related to toric varieties that will be crucial throughout the remainder of this document.

All of the toric varieties that we consider in this document are *normal*, meaning that the local ring at every point is an integrally closed domain.

1.2.1 What is a toric variety?

It is first necessary to define precisely what is meant by the "toric" part of a "toric variety", as this will be changing as one delves deeper into this dissertation. The majority of the material in this section is referenced from Cox, Little, and Schenck 2011 and Fulton 1993, and therefore only concerns toric varieties defined over $\mathbb{C}$. In the next section, we discuss the situation for toric varieties not defined over $\mathbb{C}$.

From the theory of linear algebraic groups, (see Milne 2017) we have the following definition.

Definition 1.2.1. An *affine algebraic group* is an affine variety V together with a group structure, whose binary operation $m : V \times V \to V$ is given as a morphism of varieties. The set of *algebraic maps* of two algebraic groups V and W, denoted $\mathrm{Hom}_{Alg}(V, W)$ is the set of group homomorphisms $\varphi : V \to W$ that are also morphisms of affine varieties.

Remark 1.2.1. For an affine algebraic group G defined over a field k, we have that $G := \mathrm{Spec}(R)$ with R a k-algebra. We call R the *coordinate ring* of G. The homomorphism of k-algebras induced by the map $m : G \times G \to G$ is

$$\Delta : R \to R \otimes R,$$

and is called the *comultiplication* map.

Definition 1.2.2. The *multiplicative group* $\mathbb{G}_{m,k}$ over k is represented by the coordinate ring $k[x, x^{-1}] \subset k(x)$. The comultiplication map is the k-algebra homomorphism

$$\Delta : k[x, x^{-1}] \to k[x, x^{-1}] \otimes k[x, x^{-1}] \qquad \text{such that} \qquad \Delta(x) = x \otimes x.$$

An example of an affine algebraic group that we are particularly interested in is an *algebraic torus.*

Definition 1.2.3. A *torus* T is an affine algebraic group over a field k that becomes isomorphic to a product of copies of $\mathbb{G}_m$ over a finite separable extension of k.

Example 1.2.1. With coordinates, we have that $T = (\mathbb{C}^*)^n \cong \mathbb{C}^n - V(x_1 \cdots x_n)$, where $V(f(x))$ denotes the zero-locus of the polynomial $f(x)$. The coordinate ring of the torus $(\mathbb{C}^*)^n$ is given by

$$\mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n} \cong \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] \cong \mathbb{C}[\mathbb{Z}^n].$$

The group operation $T \times T \to T$ is coordinate-wise multiplication, which can be described via the following $\mathbb{C}$-algebra homomorphism

$$\begin{gathered}\mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \to \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] \otimes \mathbb{C}[y_1^{\pm}, \ldots, y_n^{\pm}] \\ t_i \mapsto x_i \otimes y_i.\end{gathered}$$

There are two important lattices associated with a torus T, we define these now.

Definition 1.2.4. A *character* of a torus T is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. For example, $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ gives a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ given by

$$\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \cdots t_n^{a_n}$$

All characters of $(\mathbb{C}^*)^n$ arise in this way, and it follows that the characters form a group isomorphic to $\mathbb{Z}^n$. For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T.

Definition 1.2.5. A *one-parameter subgroup* of a torus T is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. For example, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by

$$\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n}).$$

All one-parameter subgroups of $(\mathbb{C}^*)^n$ arise in this way, and it follows the one-parameter subgroups form a group isomorphic to $\mathbb{Z}^n$. For an arbitrary torus T, the one-parameter subgroups form a free abelian group N of rank equal to the dimension of T.

Proposition 1.2.1. There is a natural bilinear pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ that identifies N with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and M with $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, so that M and N are dual. Additionally, we have a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong T$ via $u \otimes t \mapsto \lambda^u(t)$. In practice, the bilinear pairing is the usual dot product.

Definition 1.2.6. (Cox, Little, and Schenck 2011, Definition 1.1.3) An *affine toric variety* is an irreducible affine variety V containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an action $T_N \times V \to V$ given by a morphism.

The first chapter of Cox, Little, and Schenck 2011 outlines three equivalent methods of constructing affine toric varieties from the lattices N and M, the affine semigroup approach will be briefly discussed here.

Definition 1.2.7. A *semigroup* is a set S with an associative binary operation and an identity element. To be an *affine semigroup*, we impose the following further requirements:

- The binary operation on S is commutative. Notationally, we write this operation as "+" and the identity as "0". A finite set $\mathscr{A} \subseteq \mathsf{S}$ gives:
$$\mathbb{N}\mathscr{A} = \left\{ \sum_{m \in \mathscr{A}} a_m m \;\middle|\; a_m \in \mathbb{N} \right\} \subseteq \mathsf{S}.$$
- The semigroup is finitely generated, so that there exists a finite set $\mathscr{A} \subseteq \mathsf{S}$ such that $\mathbb{N}\mathscr{A} = \mathsf{S}$.
- The semigroup can be embedded into a lattice M.

Definition 1.2.8. Given an affine semigroup $\mathsf{S} \subset M$, the associated *semigroup algebra* is simply the vector spaces over $\mathbb{C}$ with S as a basis and multiplication induced by the semigroup structure of S. More precisely, we let M be the character lattice of a torus T_N so that $m \in M$ gives the character χ^m. Then,
$$\mathbb{C}[\mathsf{S}] = \left\{ \sum_{m \in \mathsf{S}} c_m \chi^m \;\middle|\; c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m \right\},$$
where multiplication is induced by
$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$
So, if $\mathsf{S} = \mathbb{N}\mathscr{A}$ for $\mathscr{A} = \{m_1, \dots, m_s\}$, we have that $\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$.

Proposition 1.2.2. $\operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathsf{S}$. If $\mathsf{S} = \mathbb{N}\mathscr{A}$ for a finite set $\mathscr{A} \subseteq M$, then $\operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ is the Zariski closure of the map $\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$ given by $\Phi_{\mathscr{A}}(t) = (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \in \mathbb{C}^s$.

In a similar manner, we can also describe affine toric varieties via *cones*. Fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$.

Definition 1.2.9. A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \operatorname{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \;\middle|\; \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}},$$

with $S \subset N_{\mathbb{R}}$ finite. We will say that σ is *generated* by S. Additionally, we set $\operatorname{Cone}(\emptyset) = 0$. Note also the following:

- *Convexity* in this context means that $x, y \in \sigma$ implies $\lambda x + (1 - \lambda)y \in \sigma$ for all $0 \leq \lambda \leq 1$.

- The definition of σ above is indeed a *cone*, which means that $x \in \sigma$ implies that $\lambda x \in \sigma$ for all $\lambda \geq 0$.

We shall call a polyhedral cone σ *rational* if $\sigma = \operatorname{Cone}(\mathsf{S})$ for some finite set $\mathsf{S} \subseteq N$. A desirable property that σ can possess is that the origin is a face of σ, we will call these *strictly convex* cones.

Given a polyhedral cone σ, we also have the notion of the aptly named (polyhedral) *dual cone*:

$$\sigma^\vee = \Big\{ m \in M_{\mathbb{R}} \Big\langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \Big\}.$$

From Proposition 1.2.2, we know that affine semigroups give rise to affine toric varieties. From this, we have the following Theorem.

Theorem 1.2.1. (Cox, Little, and Schenck 2011, Theorem 1.2.18)
Let $\sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ a rational polyhedral cone with associated semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_\sigma]) = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety.

Example 1.2.2. Consider the cone $\sigma_1 = \operatorname{Cone}(e_1, e_2)$ with e_1, e_2 the standard basis vectors of $\mathbb{R}^2$. A diagram of σ_1 and its dual $\sigma_1^\vee$ are shown in the figure below.

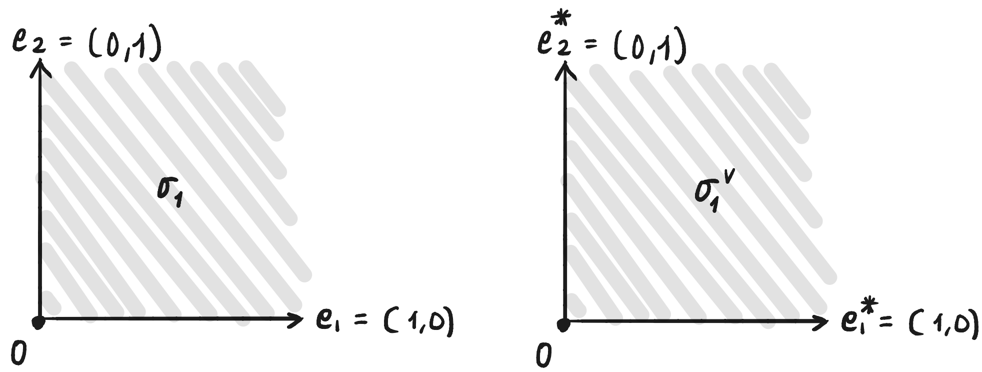

Figure 1.1 An example of a cone σ_1 and its dual $\sigma_1^\vee$.

Notice that $S_{\sigma_1} = \sigma^\vee \cap M$ is generated by $\{e_1^*, e_2^*\}$. By Theorem 1.2.1 we have

$$U_{\sigma_1} = \operatorname{Spec}(\mathbb{C}[e_1^*, e_2^*]) \cong \mathbb{C}^2.$$

Example 1.2.3. Consider the cone $\sigma_2 = \operatorname{Cone}(e_2)$ shown in Figure 1.2 below. We can see that S_{σ_2} is generated by $\{-e_1, e_1, e_2\}$. The toric variety associated to σ_2 will therefore be isomorphic to $\mathbb{C}^* \times \mathbb{C}$.

1.2.2 Fans

Now that we have seen how to construct affine toric varieties, we will discuss the construction of projective toric varieties, which brings us to the definition of a *fan*.

Figure 1.2 An example of a cone σ_2 and its dual $\sigma_2^\vee$.

Definition 1.2.10. A *fan* Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:

1. Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

2. For all $\sigma \in \Sigma$, each face of σ is also in Σ.

3. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each, and is thus also in Σ.

Notationally, we write $\Sigma(r)$ to denote the set of r-dimensional cones of Σ. The toric variety X associated to a fan Σ will be written as X_Σ.

Example 1.2.4. Perhaps the easiest example of a fan is that of $\mathbb{P}^1$, which is given by two full-dimensional cones $\sigma_1 = \text{Cone}(e_1)$ and $\sigma_2 = \text{Cone}(-e_1)$ and the trivial cone $\tau = \{0\}$. This is illustrated in Figure 1.3. The fan Σ_2 of $\mathbb{P}^1 \times \mathbb{P}^1$ is shown as well.

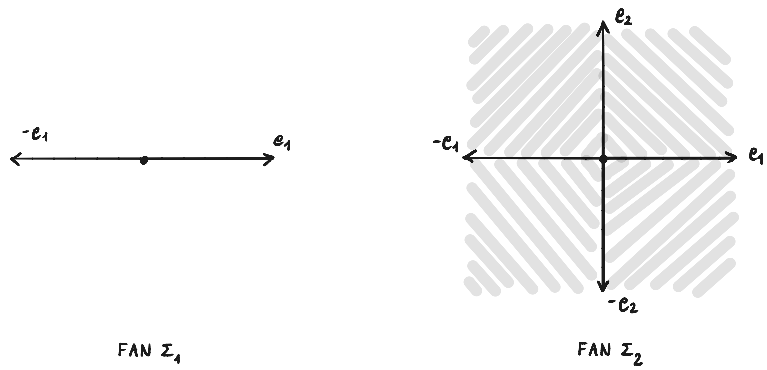

Figure 1.3 The fan Σ_1 of $\mathbb{P}^1$ and the fan Σ_2 of $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 1.2.5. A particularly nice example of (re-)constructing a variety from its associated fan is illustrated by $\mathbb{P}^2$. Writing the homogeneous coordinates of $\mathbb{P}^2$ as an ordered triple (x_0, x_1, x_2), recall that we have the following three coordinate charts:

- U_0 where $x_0 \neq 0$. We can write affine coordinates as $(x_1 x_0^{-1}, x_2 x_0^{-1}) = (z_1, z_2)$.
- U_1 where $x_1 \neq 0$. This has affine coordinates $(x_0 x_1^{-1}, x_2 x_1^{-1}) = (z_1^{-1}, z_1^{-1} z_2)$.
- U_2 where $x_2 \neq 0$. This has affine coordinates $(x_0 x_2^{-1}, x_1 x_2^{-1}) = (z_2^{-1}, z_1 z_2^{-1})$.

Consider now the fan Δ associated to $\mathbb{P}^2$, which is shown below in Figure 1.4. We can check that this is indeed a manifestation of $\mathbb{P}^2$ by looking at the dual fan $\Delta^\vee$ and reconstructing each affine piece as in the previous subsection.

Notice the following:

- S_{σ_0} is generated by $\{e_1^*, e_2^*\}$. Using variables as z_1, z_2, we see that $U_{\sigma_0} = \mathbb{C}[z_1, z_2]$.
- S_{σ_1} is generated by $\{-e_1^* + e_2, -e_1^*\}$, so that $U_{\sigma_1} = \mathbb{C}[z_1^{-1}, z_1^{-1} z_2]$.
- S_{σ_2} is generated by $\{e_1^* - e_2, -e_2^*\}$, so that $U_{\sigma_2} = \mathbb{C}[z_1 z_2^{-1}, z_2^{-1}]$.

From this computation, it is evident that $U_{\sigma_i} \cong U_i$ for $i \in \{0, 1, 2\}$. Indeed, one can check that these affine varieties glue together appropriately on the edges/rays.

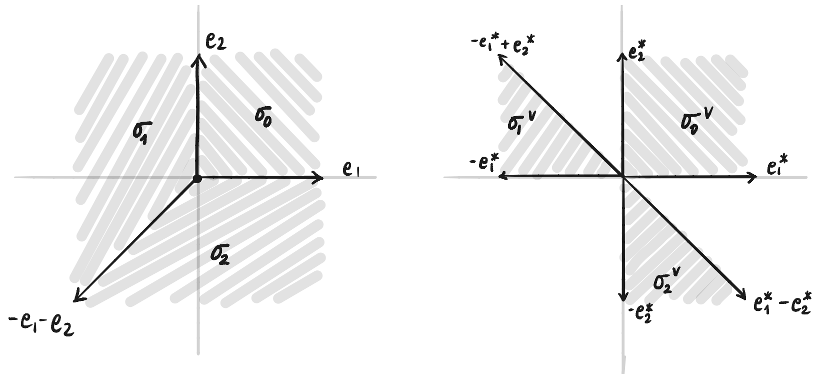

Figure 1.4 The fan Δ of $\mathbb{P}^2$ (left) and its dual (right).

While perhaps unnecessary for studying simple algebraic objects, (like projective space) the framework using cones and fans outlined here is incredibly helpful in terms of computation.

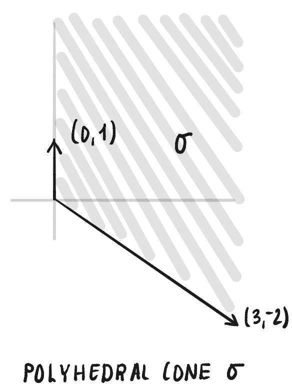

Figure 1.5 The cone σ yields an affine variety that is not smooth.

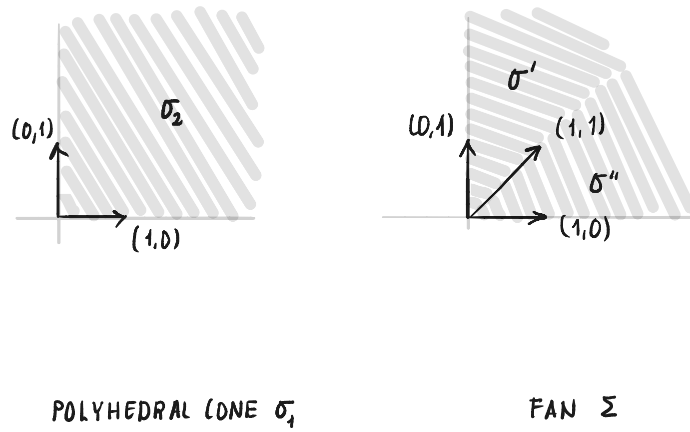

Figure 1.6 The cone σ_1 shows the variety $\mathbb{C}^2$, while the fan Σ shows $\mathbb{C}^2$ blown up at the origin.

Example 1.2.6. Given a cone σ associated to an affine toric variety X_σ, one can determine if X_σ is smooth by checking that the ray generators of σ form a subset of a basis for the lattice N. In the fan diagram shown in Figure 1.5, we can see that the variety associated with σ is not smooth, since the rays of $\sigma = \text{Cone}(e_2, 3e_1 - 2e_2)$ are not a proper subset of a set of generators for N. Another example of the utility of the fan construction is that computing and visualizing *blow-ups* becomes a matter of subdividing cones, which can be seen in Figure 1.6.

1.2.3 THE ORBIT-CONE CORRESPONDENCE

Theorem 1.2.2. (Cox, Little, and Schenck 2011, Theorem 3.2.6) Let X_Σ the toric variety of a fan $\Sigma \subset N_\mathbb{R}$, where $N_\mathbb{R}$ is of dimension $n \in \mathbb{Z}^+$. Additionally, we write $\tau \preceq \sigma$ whenever τ is a face of the cone σ. We have the following.

1. There is a bijective correspondence

$$\{\text{ cones } \sigma \text{ in } \Sigma\} \leftrightarrow \{T_N - \text{orbits in } X_\Sigma\}$$
$$\sigma \leftrightarrow O(\sigma) \simeq \text{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$$

2. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.

3. The affine open subset U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

4. $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski topology.

Corollary 1.2.1. From the Orbit-Cone correspondence, we know that $\Sigma(r)$, the set of r-dimensional cones in the fan Σ, correspond to $(n-r)$-dimensional T_N orbits in X_Σ. In particular, the set $\Sigma(1)$ gives the codimension 1 orbits in X_Σ. Given a ray $\rho \in \Sigma(1)$, the closure of its orbit is a torus invariant prime divisor on X_Σ. We denote this divisor as D_ρ.

1.2.4 Divisors

Let Σ a fan in $N_\mathbb{R}$ with $N_\mathbb{R}$ of dimension n, and X_Σ the corresponding toric variety. Note that in the remainder of this document, we are concerned only with smooth and projective varieties. For this reason, we make no distinction between Weil and Cartier divisors.

Proposition 1.2.3. (Cox, Little, and Schenck 2011, Proposition 4.1.2) For $m \in M$, the character χ^m is a rational function on X_Σ, and its divisor is given by

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

Thus, torus characters are examples of *principal divisors*.

We may now introduce the following Theorem that allows us to easily compute the Picard group for toric varieties that we are interested in.

Theorem 1.2.3. (Cox, Little, and Schenck 2011, Theorem 4.2.1) We have the following short exact sequence:

$$0 \to M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

1.2.5 The Automorphism Group of a Toric Variety

In Section 1.3 we will see the importance of the automorphism group of a variety X in classifying "forms" of X over a particular field. A this particular point, however, we have equipped ourselves with enough background to define the automorphism group of a toric variety.

Definition 1.2.11. Let X a toric variety over $\overline{F}$ with fan Σ. We define the *automorphism group* of Σ, denoted $\mathrm{Aut}(\Sigma)$, to be the group of lattice isomorphisms of N (see Definition 1.2.5) which preserves the fan Σ.

Proposition 1.2.4. (Cox 1995, Theorem 4.2, Proposition 4.5) Let X a complete toric variety with fan Σ. Then, the connected component of the identity of $\mathrm{Aut}(X)$ is generated by the torus $(\mathbb{C}^*)^{\Sigma(1)}$ and the one-parameter subgroups $y_m(\lambda)$ for $m \in R(N,\Sigma)$, where $R(N,\Sigma)$ is the set of *roots* of X given by

$$R(N,\Sigma) := \{m \in M \mid \exists \rho \in \Sigma(1) \text{ with } \langle m, n_\rho\rangle = 1 \text{ and } \langle m, n_{\rho'}\rangle \leq 0 \text{ for } \rho' \neq \rho\}.$$

Proposition 1.2.5. (Cox 1995, Corollary 4.7) For a complete simplicial toric variety X with fan Σ, $\mathrm{Aut}(X)$ is a linear algebraic group generated by T, the one-parameter subgroups $x_m(\lambda)$ for $m \in R(N,\Sigma)$, and $\mathrm{Aut}(\Sigma)$.

1.2.6 The Homogeneous Coordinate Ring

One method to study a given variety X is to study the regular functions defined on X. (See, for example, Section 1.3 of Hartshorne 1977) If X is an affine variety, this amounts to studying the *coordinate ring* of X. For toric varieties, the definition can be found in Cox 1995:

Definition 1.2.12. Let $X(\Sigma)$ be a split smooth projective toric variety associated to a fan Σ. Let R denote the *Cox ring* (or *homogeneous coordinate ring*) of $X(\Sigma)$, so that

$$R \cong k[x_\rho \mid \rho \in \Sigma(1)].$$

The Cox ring is graded by $\mathrm{Pic}(X(\Sigma))$, where the weight of x_ρ is $\mathcal{O}(D_\rho) \in \mathrm{Pic}(X(\Sigma))$. We will identify weights with elements of $\mathrm{Pic}(X(\Sigma))$.

The finite group $\mathrm{Aut}(\Sigma)$ acts via homogeneous automorphisms on R. For a weight χ and graded R-module M, we let $M(\chi)$ be the graded R-module with $M(\chi)_\psi = M_{\chi+\psi}$.

Recall that $X(\Sigma)$ is isomorphic to U modulo the Cartier dual of $\mathrm{Pic}(X(\Sigma))$ for a quasi-affine open subset U of $\operatorname{Spec} R$. As such, we have a restriction functor

$$j^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{Pic}}(\mathbb{A}^{\Sigma(1)}) \to \mathrm{D}^{\mathrm{b}}(X).$$

1.3 Arithmetic Toric Varieties

We begin by offering a slight generalization of two familiar definitions.

Definition 1.3.1. A *k-torus* is an algebraic group T over k such that $T_{\overline{k}} \cong \mathbb{G}_{m,\overline{k}}^n$ for some $n \geq 0$. We say that a torus is *split* if $T \cong \mathbb{G}_{m,k}^n$. A field extension L/k satisfying $T_L \cong \mathbb{G}_{m,L}^n$ is called a *splitting field* of the torus T. Note that any torus admits a finite Galois splitting field.

Definition 1.3.2. A *toric variety* is a normal variety X over a field k with a faithful action of an algebraic torus T with open dense orbit.

As we saw in the previous section, toric varieties over $k = \mathbb{C}$ are well-studied objects that provide a wealth of useful examples in modern algebraic geometry. However, when k is not separably closed, one does not necessarily have that T is the usual split torus $\mathbb{G}_m^n$. Further, T cannot necessarily be identified with the open orbit on which it acts, (this is always possible over $\mathbb{C}$) as the open orbit might not have any k-rational points.

Example 1.3.1. Let $X = \mathbb{P}^1_{\mathbb{R}}$. One can check that there are two nonisomorphic tori that act appropriately on $\mathbb{P}^1_{\mathbb{R}}$: the circle group $S^1 = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ via multiplication by rotation matrices, and the usual torus $\mathbb{G}_{m,\mathbb{R}} \cong \mathbb{R}^\times$. We note that S^1 and $\mathbb{G}_{m,\mathbb{R}}$ become isomorphic once base-changed to $\mathbb{C}$, since rotation matrices are diagonalizable over $\mathbb{C}$.

To fully illustrate that $S^1 \not\cong \mathbb{G}_{m,\mathbb{R}}$ and that both tori act differently upon $\mathbb{P}^1_{\mathbb{R}}$, we will explicitly write down the actions. Recall that elements of $\mathbb{P}^1_{\mathbb{R}}$ are ordered pairs $[a : b]$ with $a, b \in \mathbb{R}$ with a and b not both zero under the equivalence relation $[a : b] \sim [\lambda a : \lambda b]$ for scalars $\lambda \in \mathbb{R}^\times$.

Consider first the action of $\mathbb{G}_{m,\mathbb{R}} \cong \mathbb{R}^\times$ on $\mathbb{P}^1_{\mathbb{R}}$. For $\lambda \in \mathbb{R}^\times$, for which we simply have

$$\lambda \cdot [a : b] = [\lambda \cdot a : b],$$

so that $\mathbb{R}^\times$ acts on $\mathbb{P}^1_{\mathbb{R}}$ via multiplication in the first coordinate.

We now consider the action of $S^1 = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ on $\mathbb{P}^1_{\mathbb{R}}$. As previously mentioned, S^1 acts via rotation matrices, which in this case are 2x2 square orthogonal matrices with real entries and determinant one. For $x, y \in \mathbb{R}$ we have the following:

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot [a : b] \quad = \quad [ax - yb : bx + ya].$$

Example 1.3.2. Consider the following conic:

$$C = V\left(x^2 + y^2 + z^2\right) \subset \mathbb{P}^2_{\mathbb{R}}.$$

Notice that $C \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}}$, but $C \ncong \mathbb{P}^1_{\mathbb{R}}$ as C contains no $\mathbb{R}$-points!

Definition 1.3.3. Given a torus T, a *toric T-variety* is a normal variety with a faithful T-action and a dense open T-orbit. A toric T-variety is *split* if T is a split torus. A *splitting field* of a toric T-variety is a splitting field of T.

Definition 1.3.4. A toric T-variety X whose dense open T-orbit contains a k-rational point is called *neutral* Duncan 2016a (or a *toric T-model* Merkurjev and Panin 1997). In particular, an orbit of a split torus always has a k-point, so a split toric variety is neutral; but the converse is not true in general.

Example 1.3.3. In Example 1.3.1, we saw that for $X = \mathbb{P}^1_{\mathbb{R}}$, we can make a choice of two nonisomorphic tori acting on X. The usual torus $\mathbb{G}_m$ by definition is *split*. On the other hand, since $S^1 \ncong \mathbb{G}_m$ but does indeed admit $\mathbb{R}$-points, S^1 is *neutral* but not *split*.

1.3.1 Twists and Forms

Notice that in the examples of arithmetic toric varieties we have seen thus far, base-changing to the separable (or algebraic) closure results in a classical toric variety as described in the previous section. We have a formal definition for such a phenomenon. Let X a quasi-projective variety defined over a field k, and let F a Galois field extension of k.

Definition 1.3.5. (Poonen 2017, Definition 4.5.1) An F/k-*twist* (or F/k-*form*) of X is a k-variety Y such that there exists an isomorphism $\varphi : X_F \xrightarrow{\sim} Y_F$. A *twist* of X is a $\overline{k}/k$-twist of X.

Naturally, one might wish to classify all possible twists of a particular variety. This information is captured by a particular Galois cohomology set.

Theorem 1.3.1. (Poonen 2017, Theorem 4.5.2) In general, for a quasi-projective k-variety X and k' a Galois extension of k, the set of k-isomorphism classes of k'/k-twists of X is a pointed set with neutral element given by the isomorphism class of X. We have the following isomorphism.

$$\frac{\{k'/k\text{-twists of } X\}}{k\text{-isomorphism}} \quad \xrightarrow{\sim} \quad H^1\left(\Gamma, \mathrm{Aut}(X_{k'})\right),$$

with $\Gamma = \mathrm{Gal}(k'/k)$.

For a more detailed overview of the study of arithmetic toric varieties, the interested reader is directed to Elizondo et al. 2014a; Duncan 2016b.

1.4 Derived Categories

For the purpose of this dissertation, the precise construction of the derived category of coherent sheaves associated to a variety need not be known to the reader. Instead, we use it as a tool; one can think of the derived category as an invariant associated to a given variety. Those who are interested in the construction are directed to standard texts on the subject, such as Huybrechts 2006.

If we choose to think of the derived category as some sort of invariant associated to a variety, we run into the trouble of the unwieldily nature of such an "invariant"; it's an entire category! Perhaps even more unfortunate than the sheer *size* of this "invariant" is that it isn't even an abelian category. For this reason, one might consider ways in which the derived category can be *decomposed* into more palatable pieces. This is akin to giving a set of generators and relations to describe a group rather than describing how the group operation acts on every single group element.

Definition 1.4.1. A full triangulated subcategory of T is *admissible* if its inclusion functor admits left and right adjoints. A *semiorthogonal decomposition* of T is a sequence of admissible subcategories $\mathsf{C_1}, \ldots, \mathsf{C_s}$ such that

1. $\mathrm{Hom}_{\mathsf{T}}(A_i, A_j) = 0$ for all $A_i \in \mathrm{Obj}(\mathsf{C_i})$, $A_j \in \mathrm{Obj}(\mathsf{C_j})$ whenever $i > j$.

2. For each object T of T, there is a sequence of morphisms $0 = T_s \to \cdots \to T_0 = T$ such that the cone of $T_i \to T_{i-1}$ is an object of $\mathsf{C_i}$ for all $i \in \{1, \ldots, s\}$.

We write $\mathsf{T} = \langle \mathsf{C_1}, \ldots, \mathsf{C_s} \rangle$ to denote such a decomposition.

We consider now a particularly well-behaved example of a semiorthogonal decomposition, which provides us with an atomization that, in many ways, mimics an orthogonal decomposition of a vector space. The study of this is originally due to Beilinson 1978.

Definition 1.4.2. Let T be a k-linear triangulated category and let A be a finite dimensional k-algebra of finite homological dimension. An object E in T is A-*exceptional* if the following conditions hold:

1. $\mathrm{End}_{\mathsf{T}}(E) \cong A$.

2. $\mathrm{Ext}^n_{\mathsf{T}}(E,E) = 0$ for $n \neq 0$.

We say that E is *tilting* if it is A-exceptional for any A. We say E is *exceptional* if it is A-exceptional for a division algebra A. We say E is *étale-exceptional* if A is a finite separable field extension of k, and k*-exceptional* if A is a copy of k itself.

A totally ordered set $\mathsf{E} = \{E_1, ..., E_s\}$ of exceptional objects is an *exceptional collection* if $\mathrm{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ for all integers n whenever $i > j$. An exceptional collection is *full* if it generates T, i.e., the smallest thick subcategory of T containing E is all of T. An exceptional collection is *strong* if $\mathrm{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ whenever $n \neq 0$. An *exceptional block* is an exceptional collection $\mathsf{E} = \{E_1, ..., E_s\}$ such that $\mathrm{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ for every n whenever $i \neq j$. An exceptional collection is *étale-exceptional* if each of its objects is étale-exceptional. A collection is k*-exceptional* if each object is k-exceptional.

Remark 1.4.1. The notion of an exceptional object as defined above is present in recent literature on arithmetic applications of the derived category e.g. Ballard, Duncan, and McFaddin 2017; Ballard, Duncan, and McFaddin 2018. We note that the classical definition due to Bondal in Huybrechts 2006 still holds over $\mathbb{C}$.

Given the above definition, it is perhaps a natural question to ask exactly how "finely" a particular derived category can be decomposed. This brings us to the definition of categorical representability given in Bernardara and Bolognesi 2012a.

Definition 1.4.3. A triangulated category T is *representable in dimension m* if it admits a semiorthogonal decomposition

$$\mathsf{T} = \langle \mathsf{C}_1, \dots, \mathsf{C}_\ell \rangle$$

and for all $i = 1, \dots, \ell$ there exists a smooth projective variety Y_i with $\dim Y_i \leq m$, such that C_i is equivalent to an admissible subcategory of $\mathrm{D^b}(Y_i)$.

Let X be a smooth projective variety of dimension n. We say that X is *categorically representable* in dimension m (or equivalently in codimension $n-m$) if $\mathrm{D}^{\mathsf{b}}(X)$ is representable in dimension m.

Throughout the remainder of this text, we will explore methods in which exceptional collections and semiorthogonal decompositions of the derived category associated to a variety can be exploited to glean insight on rationality questions concerning the variety.

1.5 Noncommutative Motives

Let X be a smooth projective algebraic variety defined over a field k. An algebraic geometer who wishes to study X has a number of tools at her disposal. As we have already seen, of particular relevance to this dissertation is $\mathsf{D}^{\mathsf{b}}(X)$, the derived category of coherent sheaves on X. However, one can also associate to X many functorial invariants, such as the Grothendieck group, higher K-groups, or Hochschild homology groups. Amazingly, any correspondence between two varieties X and Y which induces an equivalence of categories between $\mathsf{D}^{\mathsf{b}}(X)$ and $\mathsf{D}^{\mathsf{b}}(Y)$ also induces an isomorphism between the functorial invariants described above. It is therefore a natural question to ask if one can completely forget about the variety X, and instead recover these functorial invariants by studying only $\mathsf{D}^{\mathsf{b}}(X)$.

The answer to this question is, in general, no. To remedy this, we enrich the derived category to get a new *dg*-category, call it $\mathcal{D}(X)$, from which we can recover $\mathsf{D}^{\mathsf{b}}(X)$ by taking the zeroth cohomology group at each complex of morphisms in $\mathcal{D}(X)$. This new category $\mathcal{D}(X)$ is the answer to our previous question: $\mathcal{D}(X)$ allows us to 'forget' about the variety X, and allows us to recover all of the aforementioned invariants by studying only $\mathcal{D}(X)$. In some sense, from the point of view of these invariants, X and $\mathcal{D}(X)$ are the same.

Example 1.5.1. Recall Beilinson's semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^n)$:

$$\mathsf{D}^{\mathsf{b}}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle.$$

Then, we have the following equivalence of dg-categories:

$$\mathcal{D}(\mathbb{P}^n) \simeq \mathcal{D}(B),$$

with $B := \mathrm{End}\left(\mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)\right)^{\mathrm{op}}$.

Example 1.5.2. In Raedschelders 2016, noncommutative motives are used to show that non-split Severi-Brauer varieties (i.e. twists of $\mathbb{P}^1$ that are *not* isomorphic to $\mathbb{P}^1$ itself) do not admit full étale exceptional collections.

We proceed now with formal definitions.

Definition 1.5.1. (see Tabuada 2015, Section 1.6.3 and Definition 4.1) To any small *dg*-category $\mathcal{A}$, one can functorially associate its *noncommutative motive*, which we will write as $\mathcal{U}(\mathcal{A})$, which takes values in a category $\mathtt{Hmo}_0(k)$. Objects of $\mathtt{Hmo}_0(k)$ are small *dg*-categories, and for any two such categories $\mathcal{A}$ and $\mathcal{B}$,

$$\mathrm{Hom}_{\mathtt{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) \cong K_0\mathtt{rep}(\mathcal{A}, \mathcal{B}),$$

where $\mathtt{rep}(\mathcal{A}, \mathcal{B})$ denotes the full triangulated subcategory of $D(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$ consisting of $\mathcal{A} - \mathcal{B}$-bimodules B such that for every $x \in \mathcal{A}$, the right $\mathcal{B}$-module $B(x, -)$ is a compact object in $D(\mathcal{B})$.

All of the details on the construction of $\mathcal{U}$ can be found in Tabuada 2005; Tabuada 2015. As in Raedschelders 2016, we need only the fact that $\mathcal{U}$ is a so-called "universal additive invariant".

Definition 1.5.2. (Tabuada 2015, Proposition 2.2) An *additive invariant* is any functor $E : \mathrm{dgcat}(k) \to D$ taking values in an additive category D such that:

1. it sends dg-Morita equivalences to isomorphisms,

2. for any pre-triangulated dg-category $\mathcal{A}$, with full pre-triangulated dg-subcategories $\mathcal{B}$ and $\mathcal{C}$ giving rise to a semi-orthogonal decomposition

$$H^0(\mathcal{A}) = \langle H^0(\mathcal{B}), H^0(\mathcal{C})\rangle,$$

   the inclusions $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{A}$ induce an isomorphism $E(\mathcal{B}) \oplus E(\mathcal{C}) \cong E(\mathcal{A})$.

1.5.1 Noncommutative Motives of Separable Algebras

We fix notation introduce the following categories as described in Tabuada and Van den Bergh 2014.

Following notation in Tabuada and Van den Bergh 2014, for a finite Γ-set S we write $k_S := \mathrm{Hom}_\Gamma(S, k_{sep})$. Let S_1, S_2 be two finite Γ-sets, and let A, B Azumaya algebras over k_{S_1} and k_{S_2} respectively. We let $\mathrm{Map}^{\Gamma,A,B}(S_1 \times S_2, \mathbb{Z})$ be the subset of $\mathrm{Map}^\Gamma(S_1 \times S_2, \mathbb{Z})$ that consists of Γ-invariant functions $\alpha : S_1 \times S_2 \to \mathbb{Z}$ such that $\alpha((s_1, s_2)) \in \mathrm{ind}_{(s_1,s_2)}(A^{op} \otimes B) \cdot \mathbb{Z}$ for every $(s_1, s_2) \in S_1 \times S_2$.

- $\mathrm{NChow}(k)$ is the category of noncommutative motives. In Tabuada 2013, this is given as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)$ consisting of smooth and proper dg categories.

- $\mathrm{CSep}(k)$ is the category of commutative separable algebras over the field k, which can also be realized as the full subcategory of $\mathrm{NChow}(k)$ consisting of objects $\mathcal{U}(A)$ for A a commutative separable k-algebra.

- $\mathrm{Sep}(k)$ is the full subcategory of $\mathrm{NChow}(k)$ consisting of objects $\mathcal{U}(A)$ for A a separable k-algebra.

- $\mathrm{CSA}(k)$ is the full subcategory of $\mathrm{Sep}(k)$ consisting of objects $\mathcal{U}(A)$ for A a central simple k-algebra. Additionally, let $\mathrm{CSA}(k)^{\oplus}$ denote the closure of $\mathrm{CSA}(k)$ under finite direct sums.

- Cov'(Γ) is the category with objects (S, A), where S is a finite Γ-set and A is an Azumaya algebra over k_S. The morphisms $\mathrm{Hom}_{\mathrm{Cov'}(\Gamma)}((S_1, A), (S_2, B))$ are the functions $\mathrm{Map}^{\Gamma, A, B}(S_1 \times S_2, \mathbb{Z})$. We have also that

  $$(S_1, A) \oplus (S_2, B) := (S_1 \sqcup S_2, A \times B) \quad (S_1, A) \otimes (S_2, B) := (S_1 \times S_2, A \otimes B),$$

  which gives Cov'(Γ) an additive structure.

Theorem 1.5.1. (Tabuada and Van den Bergh 2014, Theorem 2.12) The categories $\mathrm{Sep}(k)$ and Cov'(Γ) are equivalent.

Corollary 1.5.1. (Tabuada and Van den Bergh 2014, Corollary 2.13) There is an equivalence of categories

$$\left\{\mathcal{U}(k)^{\oplus n} \mid n \in \mathbb{N}\right\} \simeq \mathrm{CSA}(k)^{\oplus} \times_{\mathrm{Sep}(k)} \mathrm{CSep}(k).$$

In particular, from Tabuada and Van den Bergh 2013 we have that $\mathcal{U}(L)_{\mathbb{Q}} \simeq \mathcal{U}(B)_{\mathbb{Q}}$ for every finite separable field extension L/k and central simple L-algebra B.

Theorem 1.5.2. Let A an étale algebra over k, and B a separable algebra over k. In the category NCMot of noncommutative motives, if B is a summand of A, then B must be commutative. In particular, if we have central simple k_s-algebras B_s with $B := \bigotimes_s B_s$, then $\mathrm{ind}(B_s) = 1$ for all s.

Proof. Our question is whether we can find maps $f : A \to B$ and $g : B \to A$ that satisfy the following:

- $B = k_{S_2}$,

- A not commutative,

- and $g * f = 1_A$.

Notice that if $B = k_{S_2}$ then

$$\mathrm{ind}_{(s_1,s_2)}(A_{s_1}^{\mathrm{op}} \otimes B_{s_2}) = \mathrm{ind}_{s_1}(A_{s_1}^{\mathrm{op}}), \text{ and}$$

$$\mathrm{ind}_{(s_2,s_3)}(B_{s_2}^{\mathrm{op}} \otimes A_{s_3}) = \mathrm{ind}_{s_3}(A_{s_3}).$$

We'll examine $g * f(s,s)$:

$$g * f(s,s) = \sum_{s_2} f(s,s_2)g(s_2,s).$$

But notice that $f(s,s_2), g(s_2,s) \in \mathrm{ind}_s(A_s)\mathbb{Z}$, and $g * f(s,s) \in \mathrm{ind}_s(A_s)^2\mathbb{Z}$. If we want this to be the identity function (or the diagonal) we need this to be equal to 1. This cannot happen if $\mathrm{ind}_s(A_s) \neq 1$ for some s - but this happens precisely when $Z(A_s) \neq A_s$ for some factor, i.e. when A is noncommutative. From this we conclude that A cannot be an idempotent of a commutative separable algebra unless it itself is commutative. Note that this is all happening in Cov'$(G) \cong \mathrm{Sep}(k) \subset \mathrm{NChow}(k)$. $\square$

Chapter 2

On derived categories and rational points for a class of toric Fano varieties

2.1 Introduction

We provide an example of a family of arithmetic toric Fano varieties which contain an F-rational point if and only if their derived category of coherent sheaves admits a full étale exceptional collection.

2.2 A Special Class of Arithmetic Toric Varieties

We say that a toric T-variety X is *symmetric* if it has an involution $x \mapsto x'$ for which $(tx)' = t^{-1}x'$ for all $t \in T$ and $x \in X$. A complete classification of smooth, projective, symmetric toroidal Fano varieties is given in Voskresenskii and Klyachko 1984. In particular, they show the following:

Theorem 2.2.1. (Voskresenskii and Klyachko 1984, Theorem 6) A symmetric toroidal Fano variety splits into a product of projective lines and del Pezzo varieties V_{2m} for $m \in \mathbb{Z}^+$.

We present now a formal definition of this class of so-called “del Pezzo varieties”.

Definition 2.2.1. (Ballard, Duncan, and McFaddin 2018) We define the family of toric varieties denoted by V_n of dimension $n \in 2\mathbb{Z}$ as the split toric variety with rays given by

$$\begin{aligned}
e_0 &= (-1,-1,\ldots,-1) & \overline{e_0} &= (1,1,\ldots,1)\\
e_1 &= (1,0,\ldots,0) & \overline{e_1} &= (-1,0,\ldots,0)\\
e_2 &= (0,1,\ldots,0) & \overline{e_2} &= (0,-1,\ldots,0)\\
&\vdots & &\vdots\\
e_n &= (0,0,\ldots,1) & \overline{e_n} &= (0,0,\ldots,-1)
\end{aligned}$$

The maximal cones of V_n are as follows. Each maximal cone is generated by the rays in the set $\{e_i\}_{i\in A} \cup \{\overline{e_i}\}_{i\in B}$ where A and B are disjoint subsets of $\{0,1,\ldots,n\}$, each of cardinality $\frac{n}{2}$. The number of maximal cones $c(n)$ of V_n is

$$c(n) = \frac{(n+1)!}{\left(\frac{n}{2}\right)!^2}.$$

We write $\Sigma(V_n)$ to denote the fan corresponding to V_n over the separable closure. Note that V_n admits a natural $(S_2 \times S_{n+1})$-action, given by an action on the rays $e_i, \overline{e_i}$. The S_2-action, whose generator is referred to as the *antipodal involution*, is the antipodal map on the cocharacter lattice and interchanges e_i and $\overline{e_i}$ for each index i. The S_{n+1}-action permutes the indices on the rays $e_0,\ldots,e_n$ and $\overline{e_0},\ldots,\overline{e_n}$.

Example 2.2.1. When $n = 2$, the toric variety V_2 coincides with the Del Pezzo surface of degree six.

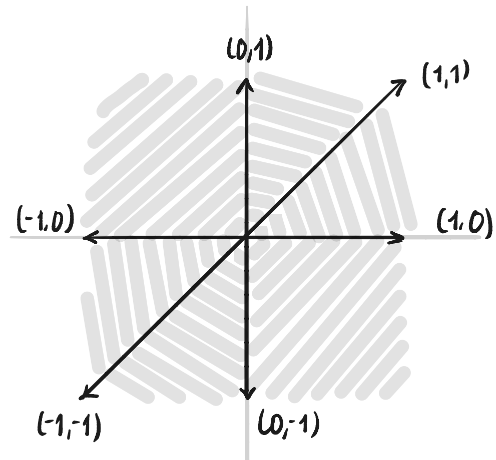

Figure 2.1 The fan Σ corresponding to the Del Pezzo surface of degree six

Remark 2.2.1. An alternative description of V_n as a blowup of $\mathbb{P}^n$ at its $(n+1)$ torus fixed points can be found in Section 3 of Casagrande 2003.

2.2.1 Divisors & A Galois-stable exceptional collection

Following the notation of Ballard, Duncan, and McFaddin 2018, we let $H, E_0, \ldots, E_n$ be the basis for $\mathrm{Pic}(V_n)$, given by the hyperplane and exceptional divisors of the blowup of $\mathbb{P}^n$ at its $(n+1)$ torus fixed points. (See Remark 2.2.1.) Then, the divisors corresponding to the rays $\{e_i\}$ and $\{\overline{e_i}\}$ are given by:

$$[e_i] = E_i, \quad [\overline{e_i}] = \left(H - \sum_{j=0}^{n} E_j\right) + E_i, \tag{2.2.1}$$

where S_{n+1} permutes the E_i, leaving H fixed, and the antipodal involution is represented by the following matrix.

$$\begin{pmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1-n & 0 & -1 & \dots & -1 \\ 1-n & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-n & -1 & -1 & \dots & 0 \end{pmatrix}$$

Now, for each $c \in \mathbb{Z}$ and $J \subset \{0, \dots, n\}$, we define:

$$F_{c,J} := c\left(\sum_{i=0}^{n} E_i - H\right) - \sum_{j \in J} E_j,$$

noting that the antipodal involution sends $F_{c,J}$ to $F_{|J|-c,J}$.

Recent work of Ballard, Duncan, and McFaddin 2018 gives a means to construct full exceptional collections collections of centrally symmetric toric Fano varieties.

Theorem 2.2.2. (Ballard, Duncan, and McFaddin 2018, Theorem 1.3) The set F_n of all bundles $\mathcal{O}(F_{c,J})$ with

1. $|J| - \frac{n}{4} \leq c \leq \frac{n}{4}$, or

2. $\frac{n+2}{4} \leq c \leq |J| - \frac{n+2}{4}$

forms a full strong $(S_{n+1} \times S_2)$-stable exceptional collection of line bundles on V_n under any ordering of the blocks such that $|J|$ is (non-strictly) decreasing.

Proposition 2.2.1. There exists a size $n+1$ block in the exceptional collection of V_n, and it is given by the pair $\left(\frac{n}{2}, n\right) \in F_n$. Additionally, for $n \geq 4$, there exists a size two block in the exceptional collection of V_n, and it is given by the pairs $(-1, 0)$ and $(1, 0) \in F_n$.

Proof. A quick application of the binomial theorem shows that the orbit corresponding to the pair $\left(\frac{n}{2}, n\right)$ will contain $n+1$ line bundles. Note also that the antipodal involution will act trivially:

$$\left(\frac{n}{2}, n\right) \mapsto \left(n - \frac{n}{2}, n\right) = \left(\frac{n}{2}, n\right).$$

From this, we see that the pair $\left(\frac{n}{2}, n\right)$ does indeed correspond to an $(S_{n+1} \times S_2)$-stable block of size $n+1$, as desired.

For the size two block, we see that the S_{n+1}-orbits corresponding to $(-1, 0)$ and $(1, 0)$ both contain one line bundle. The antipodal involution will swap the two pairs, and we therefore obtain an $(S_{n+1} \times S_2)$-stable block of size two. $\square$

2.3 Algebraically detecting rational points

Our first goal is to identify particular Brauer classes whose triviality detects the existence of rational points on a form of a generalized del Pezzo variety. From Theorem 1.3.1 we note that F-forms of V_n will be given by elements of the pointed set $H^1(F, \mathrm{Aut}_F(V_n))$, where $\mathrm{Aut}_F(V_n)$ is an algebraic group (see Duncan 2016b, Cox 1995). In order to better understand $\mathrm{Aut}_F(V_n)$, we recall Definition 1.2.11 and Propositions 1.2.4 and 1.2.5. In this case, $\mathrm{Aut}_F(V_n)$ is particularly nice- it is isomorphic to $S_{n+1} \times S_2$.

Proposition 2.3.1. For V_n, we have the following split exact sequence:

$$1 \to T \to \mathrm{Aut}_F(V_n) \to S_2 \times S_{n+1} \to 1,$$

where $S_2 \times S_{n+1} \cong \mathrm{Aut}(\Sigma(V_n))$.

Proof. We wish to show that $\mathrm{Aut}_F(V_n)$ breaks up as the direct sum of T and $S_2 \times S_{n+1}$. From Propositions 1.2.4 and 1.2.5, this amounts to showing that the set of roots $R(N, \Sigma(V_n))$ is empty. From looking at the list of rays of $\Sigma(V_n)$ in Definition 2.2.1, one can check that if such an $m \in M$ exists so that $\langle m, n_\rho \rangle = 1$, by symmetry it is always possible to choose $\rho' \neq \rho$ for which $\langle m, n'_\rho \rangle > 0$. Note that by Proposition 1.2.4, this also means that T is the connected component of the identity of $\mathrm{Aut}_F(V_n)$. $\square$

Lemma 2.3.1. A choice of étale F-algebras (K, L) (of rank 2 and $n+1$ respectively) fixes the torus of an F-form of V_n.

Proof. From Propositions 2.3.1 and 1.2.4, we have that T is the connected component of the identity of $\mathrm{Aut}_F(V_n)$, and the group of connected components is given by $S_2 \times S_{n+1}$. We note also that there is a continuous Γ-action on all of the entries of the sequence from Proposition 2.3.1. Taking cohomology of this sequence gives:

$$1 \to H^1(F, T) \to H^1\left(F, \mathrm{Aut}_F(V_n)\right) \to H^1\left(F, S_2 \times S_{n+1}\right) \to 1,$$

where $H^1(F, T)$ classifies T-torsors over F and $H^1\left(F, \mathrm{Aut}_F(V_n)\right)$ collects F-isomorphism classes of forms of V_n. Recalling that for a positive integer k, $H^1(F, S_k)$ is the set of F-isomorphism classes of degree k étale F-algebras, we see that projection onto each factor of $H^1(F, S_2 \times S_{n+1})$, yields cocycles taking values in S_2 and S_{n+1} respectively, which gives a degree 2 and a degree $(n+1)$ étale F-algebra respectively. □

Lemma 2.3.2. V_n has a F-rational point if and only if the T-torsor U is trivial.

Proof. This is an application of Proposition 4 of Voskresenskii and Klyachko 1984, which states that a smooth toric T-variety has a rational point if and only if the corresponding T-torsor U is trivial. □

Lemma 2.3.3. Fixing a torus T with character lattice $\hat{T}$, (recall from Lemma 2.3.1 that this is equivalent to making a choice of appropriate étale F-algebras (K, L)) we have from Voskresenskiĭ 1982 the following short exact sequence of Γ-modules:

$$0 \to \hat{T} \to \mathbb{Z}[KL/F] \to \mathrm{Pic}(\overline{V_n}) \to 0,$$

where we write $KL := K \otimes_F L$. Similar to the method employed for forms of the del Pezzo surface of degree 6 in Colliot-Thélène, Karpenko, and Merkurjev 2007, the short exact sequence above can be extended to:

$$0 \to \hat{T} \xrightarrow{\varphi} \mathbb{Z}[KL/F] \xrightarrow{\phi_K \oplus \phi_L} \mathbb{Z}[K/F] \oplus \mathbb{Z}[L/F] \xrightarrow{\gamma} \mathbb{Z} \to 0, \tag{2.3.1}$$

with φ, ϕ_K, and ϕ_L given by the following matrices.

$$\varphi = \begin{pmatrix} -1 & -1 & -1 & \dots & -1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix} \qquad \phi_L = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\phi_K = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}$$

Proof. Recall that $\overline{V_n}$ is an n-dimensional toric variety for $n \in 2\mathbb{Z}$, so that $\hat{T} \cong \mathbb{Z}^n$. We have also that $\mathbb{Z}[KL/F]$ is the lattice of T-invariant divisors of $\overline{V_n}$, with each basis element corresponding to a ray in the fan of $\overline{V_n}$. The map $\mathbb{Z}[KL/F] \to \mathrm{Pic}(\overline{V_n})$ sends each ray to its divisor class in $\mathrm{Pic}(\overline{V_n})$.

In equation 2.3.1, $\mathbb{Z}[L/F]$ is the lattice of pairs of opposite rays $\{e_i, \overline{e_i}\}$ for each $0 \le i \le n$, and $\mathbb{Z}[K/F]$ is the lattice of $\{e_i\} := \{e_0, \dots, e_n\}$ and $\{\overline{e_i}\} := \{\overline{e_0}, \dots, \overline{e_n}\}$.

The map $\varphi : \hat{T} \to \mathbb{Z}[KL/F]$ is given by sending the standard basis vectors of $\hat{T} \cong \mathbb{Z}^n$ to the ray generators associated to torus invariant divisors of $\overline{V_n}$. The map $\phi_L : \mathbb{Z}[KL/F] \to \mathbb{Z}[L/F]$ sends each ray to the pair containing it, and $\phi_K : \mathbb{Z}[KL/F] \to \mathbb{Z}[K/F]$ sends each ray to one of the two sets $\{e_i\}$ or $\{\overline{e_i}\}$ containing it. Finally, $\gamma : \mathbb{Z}[L/F] \oplus \mathbb{Z}[K/F] \to \mathbb{Z}$ is the difference of augmentation maps. It is then a straightforward computation to check exactness of sequence 2.3.1. $\square$

Using Lemma 2.3.3, we can write:

$$0 \to \hat{T} \to \frac{\mathbb{Z}[KL/F]}{\mathbb{Z}} \to \frac{\mathbb{Z}[K/F]}{\mathbb{Z}} \oplus \frac{\mathbb{Z}[L/F]}{\mathbb{Z}} \to 0. \tag{2.3.2}$$

Applying Cartier duality to 2.3.1 and 2.3.2 gives the following sequences of F-tori.

$$1 \to \mathbb{G}_m \to R_{K/F}(\mathbb{G}_m) \times R_{L/F}(\mathbb{G}_m) \to R_{KL/F}(\mathbb{G}_m) \to T \to 1 \tag{2.3.3}$$

$$1 \to R^{(1)}_{K/F}(\mathbb{G}_m) \times R^{(1)}_{L/F}(\mathbb{G}_m) \to R^{(1)}_{KL/F}(\mathbb{G}_m) \to T \to 1, \tag{2.3.4}$$

where $R_{E/F}(\mathbb{G}_m)$ denotes the usual Weil restriction for an extension E of F, and

$$R^{(1)}_{E/F}(\mathbb{G}_m) := \ker\Big(N_{E/F} : R_{E/F}(\mathbb{G}_m) \to \mathbb{G}_m\Big)$$

is the norm-one torus.

At this point, we are in a position to define the Brauer classes $B \in \mathrm{Br}(K)$ and $Q \in \mathrm{Br}(L)$ that will be used to detect the existence of rational points on V_n. Taking cohomology of sequence 2.3.4, the induced long exact sequence in yields the following:

$$H^1(F,T) \longrightarrow \ker\Big(\mathrm{cor}_{K/F}\Big) \times \ker\Big(\mathrm{cor}_{L/F}\Big) \quad \text{given by}$$
$$U \mapsto (\varphi_K(U), \varphi_L(U)) \in \mathrm{Br}(K) \times \mathrm{Br}(L),$$

where φ_K and φ_L denote the following connecting homomorphisms:

$$\varphi_K : H^1(F,T) \longrightarrow H^2(F, R^{(1)}_{K/F}\mathbb{G}_m) \hookrightarrow H^2(F, R_{K/F}\mathbb{G}_m) \cong \mathrm{Br}(K), \text{ and}$$

$$\varphi_L : H^1(F,T) \longrightarrow H^2(F, R^{(1)}_{L/F}\mathbb{G}_m) \hookrightarrow H^2(F, R_{L/F}\mathbb{G}_m) \cong \mathrm{Br}(L).$$

Definition 2.3.1. We define our two Brauer classes of interest to be the image of a torsor $U \in H^1(F,T)$ under the maps φ_K and φ_L. More explicitly,

$$B := \varphi_K(U) \in \mathrm{Br}(K), \quad \text{and} \quad Q := \varphi_L(U) \in \mathrm{Br}(L). \tag{2.3.5}$$

2.4 Relating Brauer Classes to Exceptional Collections

We wish to relate the definition of B and Q from the previous section to the endomorphism algebras of particular blocks of a given exceptional collection. To do so we define the algebras $\tilde{B}$ and $\tilde{Q}$ using the endomorphism algebras of particular exceptional objects, ultimately to show that in their respective Brauer groups, $[\tilde{B}] \sim B$ and $[\tilde{Q}] \sim Q$.

Fixing a pair of F-étale algebras (K, L) of the prescribed ranks, Lemma 2.3.1 gives us a torus T and a neutral toric T-model Y associated to this pair such that $\overline{Y} := Y \times_F \overline{F} \cong V_n$ for some $n \in 2\mathbb{Z}$. Fixing a torsor U of T and twisting Y by U gives $X := Y \times^T U$. From Ballard, Duncan, and McFaddin 2018, we know that $\mathsf{D}^{\mathsf{b}}(\overline{Y})$ admits a full Galois-stable exceptional collection, with orthogonal blocks E_k arranged as follows:

$$\mathsf{D}^{\mathsf{b}}(V_n) = \langle \mathsf{E}_1, \ldots, \mathsf{E}_m \rangle.$$

From Proposition 2.2.1, we know that there exist orthogonal Galois-stable blocks of size 2 and $n+1$, which we will write as $\mathsf{E}_i = \{\mathcal{L}_1, \mathcal{L}_2\}$ and $\mathsf{E}_j = \{\mathcal{J}_0, \ldots, \mathcal{J}_n\}$ respectively, with $i < j \in \{1, \ldots, m\}$.

In order to descend blocks E_i and E_j from V_n to Y, we may use Galois descent methods on the Cox rings of V_n and Y respectively, (see Definition 1.2.12) noting that if the Cox ring of V_n is $\overline{F}[x_\rho \mid \rho \in \Sigma_{V_n}(1)]$, then the Cox ring associated to Y is simply $(F[x_\rho \mid \rho \in \Sigma_{V_n}(1)])^\Gamma$. Notice that the collection given in Ballard, Duncan, and McFaddin 2018 is, by construction, stable under the action of Γ, so we need not worry about descent of sheaves from V_n to Y. However, in order to descend sheaves from V_n to X, we must ensure that the sheaves admit a T-linearization. In particular, we make use of the following Proposition from Chapter 4:

Proposition 2.4.1. (Proposition 4.3.1) For a G-equivariant sheaf (E, θ), a G-torsor U and the above isomorphism $f : X_L \to Y_L$, there is a natural isomorphism $\vartheta : f^*\Psi_U(E)_L \xrightarrow{\sim} E_L$.

Thus, to obtain exceptional blocks on the non-neutral form X of V_n, we linearize the given line bundles above with respect to T.

Recall from Equation 2.2.1 our notation conventions with respect to torus-invariant divisors of V_n. We write $x_i := [e_i]$ and $y_i := [\overline{e_i}]$ so that

$$x_i = E_i, \qquad y_i = \left(H - \sum_{j=0}^{n} E_j\right) + E_i, \tag{2.4.1}$$

and we say $E := \Sigma_{i=0}^{n} E_i$.

Lemma 2.4.1. The line bundles $\mathcal{L}_1 := H - E$ and $\mathcal{L}_2 := E - H$ (corresponding to the block E_i) each lift to $n+1$ distinct torus-invariant divisors in $\mathrm{Div}_T(\overline{X})$.

Proof. From Theorem 1.2.3 recall that we have the following short exact sequence:

$$0 \to \hat{T} \to \mathrm{Div}_T(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0.$$

We let f denote the map $\mathrm{Div}_T(\overline{X}) \to \mathrm{Pic}(\overline{X})$, noting that Sequence 1.2.3 potentially identifies many distinct torus-invariant divisors in $\mathrm{Div}_T(\overline{X})$ with a single line bundle in $\mathrm{Pic}(\overline{X})$. Because of this, we must make a choice of lift of $\mathcal{L}_1$ and $\mathcal{L}_2$ into $\mathrm{Div}_T(\overline{X})$. Notice that we may linearize $\mathcal{L}_1$ (with respect to the T-action) as $y_i - x_i$. That is, $f(y_i - x_i) = \mathcal{L}_1$. The S_{n+1}-action permutes the indices of the x_i and y_i, so that the orbit of $\mathcal{L}_1$ under S_{n+1} is the following:

$$\{y_0 - x_0, \ldots, y_i - x_i, \ldots, y_n - x_n\}.$$

Similarly, notice that the T-linearization of $\mathcal{L}_2$ under f contains $x_i - y_i$ so that the orbit of $\mathcal{L}_2$ under S_{n+1} is:

$$\{x_0 - y_0, \ldots, x_i - y_i, \ldots, x_n - y_n\}.$$

Recalling that the S_2 action swaps x_i and y_i and thus swaps $\mathcal{L}_1$ and $\mathcal{L}_2$, we see that the block

$$\{y_0 - x_0, \ldots, y_n - x_n, x_0 - y_0, \ldots, x_n - y_n\}$$

is an $S_{n+1} \times S_2$-stable block of torus-invariant divisors whose images under f are linearly equivalent to the block E_i.

□

Lemma 2.4.2. The line bundles $\mathcal{J}_i := \frac{n}{2}(E - H) - \Sigma_{k \neq i} E_k$ for $i \in \{0, \ldots, n\}$ (corresponding to the block E_j) each lift to two distinct torus-invariant divisors in $\mathrm{Div}_T(\overline{X})$.

Proof. As in the previous proof, we write $f : \mathrm{Div}_T(\overline{X}) \to \mathrm{Pic}(\overline{X})$. As torus invariant divisors, we can write each $\mathcal{J}_i$ as follows:

$$\mathcal{J}_i = \frac{n}{2}(x_i - y_i) - \sum_{j \neq i} x_j,$$

noting that the S_{n+1}-action permutes the indices on the $\mathcal{J}_i$. The S_2 action permutes the x_i and y_i, which leaves us with

$$J_i := \frac{n}{2}(y_i - x_i) - \sum_{j \neq i} y_j \in \mathrm{Div}_T(\overline{X}).$$

We show now that the image of J_i under f is $\mathcal{J}_i$. This is a straightforward computation using the notation in equation 2.4.1. Since $y_i = \left(H - \sum_{j=0}^{n} E_j\right) + E_i$, we have the following:

$$\begin{aligned}
f(J_i) &= \frac{n}{2}\left(\left(H = \sum_{j=0}^{n} E_j\right) + E_i - E_i\right) - \sum_{j \neq i}\left(\left(H - \sum_{k=0}^{n} E_k\right) + E_j\right) \\
&= \frac{n}{2}(H - E) - \sum_{j \neq i}((H - E) + E_j) \\
&= \frac{n}{2}H - \frac{n}{2}E - nH + nE + \sum_{j \neq i} E_j \\
&= \frac{n}{2}(E - H) + \sum_{j \neq i} E_j \\
&= \mathcal{J}_i.
\end{aligned}$$

Thus, we conclude $\mathcal{J}_i$ for $i \in \{0, \dots, n\}$ lifts to two distinct torus-invariant divisors:

$$\alpha_i := \frac{n}{2}(x_i - y_i) - \sum_{j \neq i} x_j, \qquad \text{and} \qquad \beta_i := \frac{n}{2}(y_i - x_i) - \sum_{j \neq i} y_j.$$

From this, we see that

$$\{\alpha_0, \beta_0, \dots, \alpha_n, \beta_n\}$$

is an $S_{n+1} \times S_2$-stable block of torus-invariant divisors whose images under f are linearly equivalent to the block E_j. $\square$

From Proposition 2.5.1 the vector bundle $\mathcal{L}_1^{\oplus n+1} \oplus \mathcal{L}_2^{\oplus n+1}$ descends to a vector bundle which we write as E_i on X, and $\mathcal{J}_1^{\oplus 2} \oplus \cdots \oplus \mathcal{J}_{n+1}^{\oplus 2}$ descends to a vector bundle denoted E_j on X.

Definition 2.4.1. We define now the following two algebras:

$$\tilde{B} := \mathrm{End}_X(E_i)^{\mathrm{op}}, \quad \text{and} \quad \tilde{Q} := \mathrm{End}_X(E_j)^{\mathrm{op}}.$$

In order to show that $[\tilde{B}]$ (resp. $[\tilde{Q}]$) is Brauer equivalent to B, (resp. Q) we need a few statements about the structure of these endomorphism algebras.

Proposition 2.4.2. $\tilde{B}$ is a rank $(n+1)^2$ Azumaya K-algebra, with $K \subset KL \subset \tilde{B}$. Similarly, $\tilde{Q}$ is a rank 4 Azumaya L-algebra with $L \subset KL \subset \tilde{Q}$.

Proof. We consider first $\tilde{B}$, which is defined to be the endomorphism algebra of the (descended) sum of two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$. Recall that $\{\mathcal{L}_1, \mathcal{L}_2\}$ forms a size two orbit in the exceptional collection given in Theorem 2.2.2. In the definition above, the claim is made that $\mathcal{L}_i^{\oplus n+1}$ for $i \in \{1, 2\}$ descends to a vector bundle on X.

Now, we show that we have embeddings $K \subset KL \subset \tilde{B}$. Note that $\mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X)^{\mathrm{op}} = F$, and since $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\mathcal{O}_X$-modules, we see that $\tilde{B}$ is an F-algebra. From the claim above, $\mathcal{L}_1^{\oplus n+1} \oplus \mathcal{L}_2^{\oplus n+1} = (\mathcal{L}_1 \oplus \mathcal{L}_2) \otimes_{\overline{F}} V$ for an $\overline{F}$ vector space of dimension $n+1$. Additionally, we see that $\mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(\mathcal{L}_2, \mathcal{L}_1) = 0$, since $\mathcal{L}_1$ and $\mathcal{L}_2$ are part of an ordered exceptional collection. We therefore have the following:

$$\begin{aligned}\mathrm{End}_{\mathcal{O}_{\overline{X}}}\left(\mathcal{L}_1^{\oplus n+1}\oplus\mathcal{L}_2^{\oplus n_1}\right) &= \mathrm{End}_{\mathcal{O}_{\overline{X}}}\left(\mathcal{L}_1\times\mathcal{L}_2\right)\otimes_{\overline{F}}\mathrm{End}_{\overline{F}}(V)\\ &= \mathrm{End}_{\overline{F}^2}\left(V_{\overline{F}^2}\right).\end{aligned}$$

Thus, we conclude that that center of $\mathrm{End}_{\mathcal{O}_{\overline{X}}}\left(\mathcal{L}_1^{\oplus n+1}\oplus\mathcal{L}_2^{\oplus n+1}\right)$ is a copy of $\overline{F}^2$, and this is contained in $\overline{F}^{2(n+1)}$. The chain $\overline{F}^2\subset\overline{F}^{2(n+1)}\subset\mathrm{End}_{\mathcal{O}_{\overline{X}}}\left(\mathcal{L}_1^{\oplus n+1}\oplus\mathcal{L}_2^{\oplus n+1}\right)$ then descends to $K\subset KL\subset\tilde{B}$, as desired.

Finally, it remains to show that $\tilde{B}$ is a rank $(n+1)^2$ Azumaya K-algebra. Let E be a separable extension of F for which the rays of $\Sigma(X)$ are defined. From Lemma 2.3.3 we see that this is equivalent to E *splitting* both K and L, i.e. $K\otimes_F E$ and $L\otimes_F E$ are finite products of F. Thus, we have that $\mathrm{End}_{\mathcal{O}_X}(E_i)\otimes_F E\cong M_{n+1}(E^2)$ with $E^2=K\otimes_F E$, and we conclude that $\tilde{B}$ is indeed a rank $(n+1)^2$ Azumaya K-algebra containing a copy of KL as a subalgebra.

A nearly identical (up to any necessary rank changes) proof can be used to show that $\tilde{Q}$ is a rank four Azumaya L-algebra, with $L\subset KL\subset\tilde{Q}$. □

Proposition 2.4.3. $[B]\sim[\tilde{B}]$ in $\mathrm{Br}(K)$, and $[Q]\sim[\tilde{Q}]$ in $\mathrm{Br}(L)$.

Proof. Let (K,L) a fixed pair of F-étale algebras of the prescribed ranks. This choice of étale algebras gives rise to a neutral toric T-model Y associated to this pair such that $\overline{Y}:=Y\times_F\overline{F}\cong V_n$ for some $n\in 2\mathbb{Z}$. Fixing a torsor U of T and twisting Y by U gives $X:=Y\times^T U$, a non-neutral form of V_n. Definition 2.4.1 gives us two algebras associated with Y, which we call $\tilde{B}_Y$ and $\tilde{Q}_Y$. Similarly, we have two algebras associated to X, which we call $\tilde{B}$ and $\tilde{Q}$.

Recall the exact sequence from Equation 2.3.2:

$$0\to\hat{T}\to\frac{\mathbb{Z}[KL/F]}{\mathbb{Z}}\to\frac{\mathbb{Z}[K/F]}{\mathbb{Z}}\oplus\frac{\mathbb{Z}[L/F]}{\mathbb{Z}}\to 0.$$

Modding out by $\mathbb{Z}[L/F]/\mathbb{Z}$ and $\mathbb{Z}[K/F]/\mathbb{Z}$ respectively gives us:

$$0\to\hat{T}\to\left(\frac{\mathbb{Z}[KL/F]}{\mathbb{Z}}\right)\Big/\frac{\mathbb{Z}[L/F]}{\mathbb{Z}}\to\frac{\mathbb{Z}[K/F]}{\mathbb{Z}}\to 0. \tag{2.4.2}$$

$$0 \to \hat{T} \to \left(\frac{\mathbb{Z}[KL/F]}{\mathbb{Z}} \right) \Big/ \frac{\mathbb{Z}[K/F]}{\mathbb{Z}} \to \frac{\mathbb{Z}[L/F]}{\mathbb{Z}} \to 0. \tag{2.4.3}$$

Applying Cartier duality to Sequence 2.4.2 gives the following sequences of F-tori:

$$1 \to R^{(1)}_{K/F}(\mathbb{G}_m) \to \mathcal{G}_L \to T \to 1, \tag{2.4.4}$$

and

$$1 \to R^{(1)}_{L/F}(\mathbb{G}_m) \to \mathcal{G}_K \to T \to 1, \tag{2.4.5}$$

where $\mathcal{G}_L$ and $\mathcal{G}_K$ are given by

$$\begin{aligned} \mathcal{G}_L &:= \operatorname{Ker}\left(N_{KL/L} : R_{KL/F}(\mathbb{G}_m) \to R_{L/F}(\mathbb{G}_m)\right), \\ \mathcal{G}_K &:= \operatorname{Ker}\left(N_{KL/K} : R_{KL/F}(\mathbb{G}_m) \to R_{K/F}(\mathbb{G}_m)\right). \end{aligned}$$

Equation 2.4.4 (resp. 2.4.5) together with our embedding of $K \subset KL \subset \tilde{B}_Y$ (resp. $L \subset KL \subset \tilde{Q}_Y$) given in Proposition 2.4.2 induce the following commutative diagrams:

$$\begin{tikzcd} 1 \arrow[r] & R^{(1)}_{K/F}(\mathbb{G}_m) \arrow[r] \arrow[d] & \mathcal{G}_L \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & R_{K/F}(\mathbb{G}_m) \arrow[r] & R_{K/F}(\mathrm{GL}(\tilde{B}_Y)) \arrow[r] & R_{K/F}(\mathrm{PGL}(\tilde{B}_Y)) \arrow[r] & 1 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r] & R^{(1)}_{L/F}(\mathbb{G}_m) \arrow[r] \arrow[d] & \mathcal{G}_K \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & R_{L/F}(\mathbb{G}_m) \arrow[r] & R_{L/F}(\mathrm{GL}(\tilde{Q}_Y)) \arrow[r] & R_{L/F}(\mathrm{PGL}(\tilde{Q}_Y)) \arrow[r] & 1 \end{tikzcd}$$

where the vertical arrows map the top F-tori diagonally into the entires of the bottom rows. This induces the following cohomology diagram, and a similar diagram for $\tilde{Q}_Y$.

(It may also be helpful to note that one can rewrite the cohomology sets involving Weil restrictions by applying *Shaprio's Lemma*, see for example Lemma 1.4.5 of Poonen 2017)

$$\begin{tikzcd} & H^1(F,T) \arrow[r] \arrow[d] & \ker\left(\mathrm{cor}_{K/F} : \mathrm{Br}(K) \to \mathrm{Br}(F)\right) \arrow[d] \\ 1 \arrow[r] & H^1(K, \mathrm{PGL}(\tilde{B}_Y)) \arrow[r] & \mathrm{Br}(K) \end{tikzcd}$$

The map $H^1(F,T) \to H^1(K, \mathrm{PGL}(\tilde{B}_Y))$ is given by sending a T-torsor U to the endomorphism algebra $\tilde{B}$, the top horizontal map $H^1(F,T) \to \ker(\mathrm{cor}_{K/F})$ is the map φ_K from Definition 2.3.1, the bottom horizontal map sends a K-algebra B' to $[B'] \in \mathrm{Br}(K)$, and the right-most vertical homomorphism is inclusion. Commutativity of this diagram shows us that we do indeed have $[\tilde{B}] \sim B$, as required. A similar argument can be used to show that $[\tilde{Q}] \sim Q$ in $\mathrm{Br}(L)$.

□

Lemma 2.4.3. The splitting of the Azumaya K-algebra $\tilde{B}$ as well as the Azumaya L-algebra $\tilde{Q}$) detect the triviality of the T-torsor. More explicitly, The T-torsor U is is trivial if and only if both $\tilde{B}$ and $\tilde{Q}$ are split.

Proof. With Lemma 2.4.3, it is sufficient to observe that

$$H^1(F,T) \to \ker\left(\mathrm{cor}_{K/F}\right) \times \ker\left(\mathrm{cor}_{L/F}\right)$$

is injective in order to prove the claim. □

2.5 Main Theorem

The goal of this section is to show the main theorem of this chapter. Before we proceed to the proof, we make some useful observations about the exceptional collection of V_n. While the exceptional collection given in Theorem 2.2.2 is Galois-stable, it also satisfies a stronger condition: it is of *TCI-type*.

(This notion will appear again in Chapter 4.) In order to define this, we recall the definition of the *Cox ring* (or *homogeneous coordinate ring)* associated to a toric variety from Definition 1.2.12. Let R denote the Cox ring associated to the variety $X(\Sigma)$.

Definition 2.5.1. We say $X(\Sigma)$ has an exceptional collection of *TCI-type* if there exists a set of graded R-modules $F_1, \ldots, F_t$ such that

- for each $1 \leq s \leq t$
$$F_s = R(\chi_s)/(x_l \mid l \in I_s)$$
for some $\chi_s \in \mathrm{Pic}(X(\Sigma))$ and $I_s \subseteq \Sigma(1)$,
- the set $F_1, \ldots, F_t$ is $\mathrm{Aut}(\Sigma)$-stable, and
- the set of isomorphism classes of $j^*F_1, \ldots, j^*F_t$ forms a k-exceptional collection of $\mathrm{D^b}(X(\Sigma))$.

Proposition 2.5.1. Let $X(\Sigma)$ be a split smooth projective toric variety over k with fan Σ and X a neutral smooth projective toric T-variety such that $X(\Sigma)_L \cong X_L$ for some extension L/k. If $X(\Sigma)$ posseses a full exceptional collection of TCI-type, then X possesses a full étale exceptional collection such that each object is T-linearizable up to taking sums.

Proof. This can be found in the proof of Proposition 4.4.2. □

Lemma 2.5.1. The collection given in Theorem 2.2.2 is of TCI-type.

Proof. Let $\mathsf{D^b}(V_n) = \langle \mathsf{E}_1, \ldots, \mathsf{E}_k \rangle$ be the collection of exceptional blocks given in Theorem 2.2.2, with E_i and E_j the blocks corresponding to the endomorphism algebras defined in Definition 2.4.1.

Each block E_s with $s \in \{1, \ldots, k\}$ consists of line bundles, which lift to $\mathrm{Spec}(R)$ in an $\mathrm{Aut}(\Sigma)$-stable manner without issue due to Ballard, Duncan, and McFaddin 2018. We note also that Lemmas 2.4.1 and 2.4.2 reveal that the orbits of the stabilizers of blocks E_i and E_j in $\mathrm{Div}_T(V_n)$ give Azumaya algebras of ranks $(n+1)^2$ and 4, respectively. □

Theorem 2.5.1. The arithmetic toric variety V_n has an F-rational point if and only if $\mathsf{D}^{\mathsf{b}}(V_n)$ admits a full étale exceptional collection.

Proof. Let X be a neutral form of V_n over F. (Recall from Chapter 1.3 that this means X has an F-rational point.) By Lemma 2.3.2, we know that X has an F-rational point if and only if the T-torsor U is trivial, and from Lemma 2.4.3 we know that this occurs if and only if the algebras $\tilde{B}$ and $\tilde{Q}$ associated to X are split.

From Theorem 2.2.2, $\mathsf{D}^{\mathsf{b}}\left(\overline{X}\right)$ (over $\overline{F}$) admits the following full exceptional collection:

$$\mathsf{D}^{\mathsf{b}}(\overline{X}) = \mathsf{D}^{\mathsf{b}}(V_n) = \langle \mathsf{E}_1, \ldots, \mathsf{E}_k \rangle,$$

where each block E_i descends to an exceptional object E_i for $\mathsf{D}^{\mathsf{b}}(X)$. Of particular interest are the descended vector bundles E_i and E_j associated to the algebras $\tilde{B}$ and $\tilde{Q}$ from Definition 2.4.1.

From Chapter 1.4 and Tabuada 2015, we know that applying the universal additive invariant $\mathcal{U}$ gives us the following:

$$\mathcal{U}\left(\mathsf{D}^{\mathsf{b}}(X)\right) = \mathcal{U}(\mathcal{O}) \oplus \mathcal{U}(E_1) \oplus \cdots \oplus \mathcal{U}(E_i) \oplus \cdots \oplus \mathcal{U}(E_j) \oplus \cdots \oplus \mathcal{U}(E_k),$$

Where $\mathcal{U}(E_s) = \mathrm{End}(E_s)$ for $s \in \{1, \ldots, k\}$. Since $\tilde{B}$ and $\tilde{Q}$ are assumed to be split, we see that $\mathcal{U}(E_i) \simeq \mathcal{U}(\tilde{B}) \simeq \mathcal{U}(M_n(K))$, and similarly for E_j. Morita invariance of $\mathcal{U}$ implies $\mathcal{U}(M_n(K)) \simeq \mathcal{U}(K)$. Since the collection from Theorem 2.2.2 is of TCI-type we may conclude using Lemma 2.5.1 that the endomorphism algebra of each exceptional object in our collection is étale. Thus, $\mathsf{D}^{\mathsf{b}}(X)$ does indeed admit a full étale exceptional collection.

Suppose now that a form Y of V_n is such that $\mathsf{D}^{\mathsf{b}}(Y)$ admits a full étale exceptional collection giving rise to a semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(Y) = \langle \mathsf{A}_1, \ldots, \mathsf{A}_s \rangle$. Section 2.2 together with the additivity of $\mathcal{U}(-)$ on semiorthogonal decompositions gives the following isomorphisms of noncommutative motives:

$$\begin{aligned} \mathcal{U}(\mathcal{O}) \oplus \mathcal{U}(A_1) \oplus \cdots \oplus \mathcal{U}(A_s) &\simeq \mathcal{U}\left(\mathsf{D}^{\mathsf{b}}(Y))\right) \\ &\simeq \mathcal{U}(\mathcal{O}) \oplus \mathcal{U}(E_1) \oplus \cdots \oplus \mathcal{U}(E_k), \end{aligned} \tag{2.5.1}$$

where the last isomorphism comes from descending the exceptional collection from Theorem 2.2.2. Theorem 1.5.2 guarantees that the length of the sums on either side of the isomorphism in equation 2.5.1 are equal, that is: $s = k$. By assumption, $\mathsf{D}^{\mathsf{b}}(Y)$ admits a full étale exceptional collection, so that each $\mathcal{U}(A_i)$ is an étale algebra over the base field. From this, we conclude that B and Q are indeed split.

□

Chapter 3

Separable Algebras and Coflasque Resolutions

This chapter appears in joint work with Dr. Matthew Robert Ballard, Dr. Alexander Duncan, and Dr. Patrick McFaddin.

3.1 Introduction

Given a base field k, an n-dimensional *Severi-Brauer variety* X over k is an (étale) k-form of the projective space $\mathbb{P}^n_k$; in other words, there exists a finite separable field extension L/k such that $X_L := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ is isomorphic to $\mathbb{P}^n_L$. The isomorphism classes of Severi-Brauer varieties of dimension n are in bijective correspondence with central simple algebras A of degree $n+1$, which are forms of the algebra $M_{n+1}(k)$ of $(n+1) \times (n+1)$ matrices over k.

From another perspective, the isomorphism classes of n-dimensional Severi-Brauer varieties over k are classified by the elements of the Galois cohomology set $H^1(k, \mathrm{PGL}_{n+1})$. There is an injective function

$$H^1(k, \mathrm{PGL}_n) \hookrightarrow \mathrm{Br}(k) = H^2(k, \mathbb{G}_m),$$

functorial with respect to the field k, which associates a given Severi-Brauer variety to the Brauer equivalence class of the corresponding central simple algebra.

A *separable algebra* A over a field k is a direct sum

$$A = \bigoplus_{i=1}^{r} M_{n_i}(D_i)$$

of matrix algebras $M_{n_i}(D_i)$ where each D_i is a division k-algebra whose center is a separable field extension of k. Alternatively, a separable algebra is an étale k-form of a direct sum of matrix algebras over k. The del Pezzo surfaces of degree 6 are all k-forms of one another. Blunk 2010 demonstrated how to associate a form of a separable k-algebra to each del Pezzo surface of degree 6 in such a way that two surfaces are isomorphic if and only if their corresponding algebras are isomorphic. In this case, the split del Pezzo surface has the associated separable algebra $M_2(k)^{\oplus 3} \oplus M_3(k)^{\oplus 2}$.

Both Severi-Brauer varieties and del Pezzo surfaces are examples of *arithmetic toric varieties*: normal varieties which admit a faithful action of a torus (Definition 3.2.3) with dense open orbit. In Duncan 2016a, it is shown that one can distinguish isomorphism classes of k-forms of an arithmetic toric variety X by separable k-algebras whenever forms of X with a rational point are retract rational. In all these cases, the separable algebras are the direct sums of endomorphism algebras of certain indecomposable vector bundles on the variety X.

It is natural to ask: can one can distinguish k-forms for wider classes of objects via separable algebras? For example, can varieties be distinguished by endomorphism algebras of exceptional objects in their derived categories? Are there even more exotic constructions? The purpose of this paper is to precisely describe a fundamental obstruction to all such strategies.

Recall that, under mild technical conditions, the isomorphism classes of k-forms of an algebraic object X are in bijection with the Galois cohomology set $H^1(k, G)$, where G is the automorphism group scheme of X. If A is a separable k-algebra with an algebraic action of an algebraic group G, then there is an algebraic group homomorphism $G \to \mathrm{Aut}(A)$. We define

$$\text{Ж}(k, G) := \bigcap_A \ker \left(H^1(k, G) \to H^1(k, \mathrm{Aut}(A)) \right) \tag{3.1.1}$$

where the intersection runs over all separable k-algebras A with a G-action. Informally, $\text{Ж}(k, G)$ is the set of k-forms of X that cannot be distinguished using forms of separable k-algebras.

Our main result completely characterizes this invariant using the theory of flasque and coflasque resolutions of reductive algebraic groups, pioneered in Colliot-Thélène 2004; Colliot-Thélène 2008, which we review in Section 3.2 below.

Theorem 1. Let G be a (connected) reductive algebraic group over k. Then

$$\text{Ж}(k, G) = \mathrm{im} \left(H^1(k, C) \to H^1(k, G) \right),$$

where

$$1 \to S \to C \to G \to 1$$

is any coflasque resolution of G of the second type.

Remark 3.1.1. The reductive hypothesis is harmless in characteristic 0, since in this case there is a canonical isomorphism $H^1(k, G) \cong H^1(k, G/U)$ for a connected linear algebraic group G with unipotent radical U.

Remark 3.1.2. For a finite constant group G, the invariant $\text{Ж}(k, G)$ is always trivial for almost tautological reasons. Indeed, $H^1(k, G)$ classifies G-Galois algebras over k, which are, in particular, separable algebras with a G-action.

Remark 3.1.3. The automorphism group scheme of a toric variety is not usually connected. Indeed, this is not even true for del Pezzo surfaces of degree 6 studied in Blunk 2010. However, using non-abelian H^2 as in Duncan 2016a, one sees that Theorem 1 is sufficient for most purposes. In particular, Theorem 1 shows that the strategy for distinguishing forms of toric varieties using separable algebras in Duncan 2016a is essentially the best one can expect.

Remark 3.1.4. Our initial motivation for introducing $\maltese(k, G)$ was to find examples of pairs of k-forms of varieties that are derived-equivalent but not isomorphic. If the derived category of a G-variety X has a G-stable exceptional collection $\{E_1, \ldots, E_n\}$, then the direct sum $A_X := \bigoplus_{i=1}^n \mathrm{End}(E_i)$ is a separable algebra with a G-action (see Ballard, Duncan, and McFaddin 2017). However, $B_X := \mathrm{End}(\bigoplus_{i=1}^n E_i)$ is not a separable algebra in general and A_X is only its semisimplification. In particular, we do not allow *arbitrary* finite-dimensional associative k-algebras A in the definition (3.1.1) of $\maltese(k, G)$.

3.1.1 Cohomological invariants

Recall that the Galois cohomology pointed set $H^i(k, G)$ is functorial in both G and k. In particular, fixing G, we may view $H^i(-, G)$ as a functor from the category of field extensions of k to the category of pointed sets (or to groups, or to abelian groups, appropriately). Let $\mathrm{Inv}^2_*(G, S)$ be the group of *normalized cohomological invariants*, i.e., natural transformations

$$\alpha : H^1(-, G) \to H^2(-, S)$$

where G is a linear algebraic group, S is a commutative linear algebraic group, and α takes the distinuished point to zero.

Recall that a linear algebraic group G is *special* if $H^1(K, G_K)$ is trivial for all field extensions K/k. In Theorem 3.4.1 below, we will see that for reductive algebraic groups G we have many equivalent characterizations of $\maltese(k, G)$. In particular,

$$\maltese(k, G) = \bigcap_S \bigcap_\alpha \ker \left(\alpha(k) \colon H^1(k, G) \to H^2(k, S) \right)$$

where the intersections run over all special tori S and all normalized cohomological invariants $\alpha \in \mathrm{Inv}^2_*(G, S)$.

Thus, not only is $Ж(k, G)$ an obstruction to differentiating Brauer classes obtained from actions of G on separable algebras, but also to those obtained from completely arbitrary maps (provided they behave well under field extensions). In order to demonstrate this, we prove the following, which may be of independent interest.

Theorem 2. Let G be a reductive algebraic group over k and S a special torus over k. Let $\mathrm{Ext}^1_k(G, S)$ be the group of isomorphism classes of central algebraic group extensions of G by S under Baer sum. Then the canonical map

$$\mathrm{Ext}^1_k(G, S) \to \mathrm{Inv}^2_*(G, S)$$

that takes an extension ξ to its connecting homomorphism ∂_ξ, is an isomorphism of groups.

The above theorem is a generalization of a result from Blinstein and Merkurjev 2013, Theorem 2.4 showing that there *exists* an isomorphism when $S = \mathbb{G}_m$. However, our result is stronger even when $S = \mathbb{G}_m$ since it proves that this specific map is an isomorphism.

3.1.2 Applications

Theorem 1 allows us to compute $Ж(k, G)$ in many cases of interest. For example, the following consequences are discussed in Section 3.5:

- when k is a finite field or nonarchimedean local field $Ж(k, G)$ is trivial.
- if S is a torus, the functor $Ж(-, S)$ is trivial if and only if S is retract rational.
- if S is a torus over a number field k, then

$$Ж(k, S) = Ш^1(k, S).$$

- if G is semisimple and simply-connected over a number field, then

$$Ж(k, G) = \prod_{v \text{ real}} Ж(k_v, G_v).$$

- if k is a totally imaginary number field, then

$$\text{Ж}(k, G) = \text{Ш}^1(k, G).$$

Retract rationality will be recalled in Section 3.2 below (see Definition 4.2.3) and $\text{Ш}^1(k, G)$ denotes the Tate-Shafarevich group, which is discussed in Section 3.5. Indeed the notation Ж was chosen to remind the reader of Ш. The connection is made explicit for number fields in the following:

Theorem 3. Let G be a reductive algebraic group over a number field k. Then there exists a canonical isomorphism

$$\text{Ж}(k, G) = \text{Ш}^1(k, G) \times \prod_{v \text{ real}} \text{Ж}(k_v, G_{k_v}).$$

The structure of the remainder of the paper is as follows. In Section 3.2, we overview the theory of coflasque and flasque resolutions, moving from lattices to tori and then treating general reductive algebraic groups. In Section 3.3, we review cohomological invariants and prove Theorem 2. In Section 3.4, we prove Theorem 1 as well as several other equivalent characterizations of $\text{Ж}(k, G)$. Finally, in Section 3.5, we compute $\text{Ж}(k, G)$ in several special cases and establish Theorem 3.

Acknowledgements

The authors would like to thank B. Antieau for several helpful comments. Via the first author, this material is based upon work supported by the National Science Foundation under Grant No. NSF DMS-1501813. Via the second author, this work was supported by a grant from the Simons Foundation (638961, AD). The third author was partially supported by a USC SPARC grant. The fourth author was partially supported by an AMS-Simons travel grant.

Notation and Conventions

Throughout, k denotes an arbitrary field with separable closure $\overline{k}$. Let Γ_k denote the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$, which is a profinite group. A variety is an integral separated scheme of finite type over a field. A linear algebraic group is a smooth affine group scheme of finite-type over k. A reductive group is assumed to be connected.

Let $\pi : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ be the morphism associated to a separable field extension L/k. For a k-variety X, we write $X_L := X \times_{\mathrm{Spec}\, L} \mathrm{Spec}\, k = \pi^*(X)$ and $\overline{X} := X_{\overline{k}}$. For an L-variety Y, we write $R_{L/k}(Y) := \pi_*(Y)$ for the Weil restriction, which is a k-variety.

Let GL_n denote the general linear group scheme and $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t^{\pm 1}]) = \mathrm{GL}_1$ as the multiplicative group over $\mathbb{Z}$. We will simply write GL_n for $\mathrm{GL}_{n,k}$ or $\mathbb{G}_m$ for $\mathbb{G}_{m,k}$ when there is no danger of confusion. Unless otherwise specified, a G-torsor is a *right* G-torsor.

We will reference the following categories:

- Set is the category of sets.
- Set_* is the category of pointed sets.
- Grp is the category of groups.
- Ab is the category of abelian groups.
- Lat is the category of finitely-generated free abelian groups.

Given a base field k:

- k-Alg is the category of associative k-algebras.
- k-Fld is the category of field extensions of k.
- k-Grp is the category of algebraic groups over k.

Given a profinite group Γ and a concrete category C (in other words, C is equipped with a faithful functor to category of sets), we write Γ-C to denote the category of objects whose underlying sets are endowed with the discrete topology and a continuous left action of Γ. Objects in Γ-Set, Γ-Ab, and Γ-Lat are called Γ-sets, Γ-modules, and Γ-lattices respectively.

For Γ-modules A, B, we use the shorthand notation $\mathrm{Hom}_\Gamma(A, B) := \mathrm{Hom}_{\Gamma\text{-}\mathsf{Ab}}(A, B)$ and $\mathrm{Ext}^i_\Gamma(A, B) := \mathrm{Ext}^i_{\Gamma\text{-}\mathsf{Ab}}(A, B)$. For linear algebraic groups A, B over k with B commutative, we denote by $\mathrm{Ext}^1_k(A, B)$ the group of isomorphism classes of central extensions of algebraic groups

$$1 \to B \to G \to A \to 1$$

under the usual Baer sum.

For k-algebras A and B, we use the shorthand $\mathrm{Hom}_k(A, B) := \mathrm{Hom}_{k\text{-}\mathsf{Alg}}(A, B)$. For algebraic groups A and B defined over k, we use the shorthand $\mathrm{Hom}_k(A, B) := \mathrm{Hom}_{k\text{-}\mathsf{Grp}}(A, B)$. For a scheme X and an étale sheaf $\mathcal{F}$ on X, we write $H^n(X, \mathcal{F})$ to denote étale cohomology. In particular, we write $\mathrm{Pic}(X) = H^1(X, \mathbb{G}_m)$ and $\mathrm{Br}(X) = H^2(X, \mathbb{G}_m)$. For a field k, we write $H^n(k, \mathcal{F}) := H^n(\mathrm{Spec}(k), \mathcal{F})$. For a profinite group Γ and a (continuous) Γ-set A, we write $H^n(\Gamma, A)$ for the appropriate cohomology set, assuming this makes sense given n and A.

3.2 Coflasque resolutions

3.2.1 Preliminaries on lattices

We recall some (mostly standard) facts about Γ-lattices; see, for example, Colliot-Thélène and Sansuc 1977 or Voskresenskiĭ 1998.

Definition 3.2.1. Let Γ be a profinite group and let M be a Γ-lattice. Note that the image of the Γ-action factors through a finite group G called the *decomposition group*, which acts faithfully on M.

1. M is *permutation* if there is a $\mathbb{Z}$-basis of M permuted by Γ.

2. M is *stably permutation* if there exist permutation lattices P_1 and P_2 such that $M \oplus P_1 = P_2$.

3. M is *invertible* if it is a direct summand of a permutation lattice.

4. M is *quasi-permutation* if there exists a short exact sequence

$$0 \to M \to P_1 \to P_2 \to 0$$

where P_1 and P_2 are permutation lattices.

Given a Γ-lattice M, let $[M]$ denote its similarity class. In other words, $[M_1] = [M_2]$ if and only if there exist permutation Γ-lattices P_1 and P_2 such that $M_1 \oplus P_1 \cong M_2 \oplus P_2$. Observe that the set of similarity classes form a monoid under direct sum. Being stably permutation amounts to saying that $[M] = [0]$, while being invertible amounts to saying there exists a lattice L such that $[M] + [L] = [0]$.

Given a Γ-lattice M, the *dual lattice* $M^\vee := \mathrm{Hom}_{\mathsf{Ab}}(M, \mathbb{Z})$ is the set of group homomorphisms from M to $\mathbb{Z}$ with the natural Γ-action where $\mathbb{Z}$ has the trivial Γ-action. Note that this duality induces an exact anti-equivalence of the category of Γ-lattices with itself.

Definition 3.2.2. Let M be a Γ-lattice.

1. M is *coflasque* if $H^1(\Gamma', M) = 0$ for all open subgroups $\Gamma' \subseteq \Gamma$.

2. M is *flasque* if $M^\vee$ is coflasque.

3. A *flasque resolution of M of the first type* is an exact sequence

$$0 \to M \to P \to F \to 0$$

while a *flasque resolution of M of the second type* is an exact sequence

$$0 \to P \to F \to M \to 0$$

where, in each case, P is a permutation lattice and F is a flasque lattice.

4. A *coflasque resolution of M of the first type* is an exact sequence

$$0 \to C \to P \to M \to 0$$

while a *coflasque resolution of M of the second type* is an exact sequence

$$0 \to M \to C \to P \to 0$$

where, in each case, P is a permutation lattice and C is a coflasque lattice.

The following alternative characterizations of flasque, coflasque, and invertible will be useful:

Lemma 3.2.1. Let Γ be a profinite group.

1. The following are equivalent for a Γ-module C:

    - C is coflasque.
    - $\mathrm{Ext}^1_\Gamma(P, C) = 0$ for every permutation Γ-lattice P.
    - $\mathrm{Ext}^1_\Gamma(Q, C) = 0$ for every invertible Γ-lattice Q.

2. The following are equivalent for a Γ-module F:

    - F is flasque.
    - $\mathrm{Ext}^1_\Gamma(F, P) = 0$ for every permutation Γ-lattice P.
    - $\mathrm{Ext}^1_\Gamma(F, Q) = 0$ for every invertible Γ-lattice Q.

3. The following are equivalence for a Γ-module M:

    - M is invertible.
    - $\mathrm{Ext}^1_\Gamma(M, C) = 0$ for every coflasque Γ-lattice C.
    - $\mathrm{Ext}^1_\Gamma(F, M) = 0$ for every flasque Γ-lattice F.

Proof. This is standard. See, e.g., Colliot-Thélène and Sansuc 1977, Lemme 9 and Colliot-Thélène and Sansuc 1987, p. 0.5. □

Flasque/coflasque resolutions of both types always exist but are never unique; however, the similarity classes $[F]$ and $[C]$ are well-defined Colliot-Thélène and Sansuc 1987, Lemma 0.6.

It is well known that flasque and coflasque resolutions of the first type are "versal" in the following sense:

Lemma 3.2.2. Let M be a Γ-lattice. If

$$0 \to C \to P \xrightarrow{\alpha} M \to 0$$

is a coflasque resolution of the first type, then any morphism $P' \to M$ with P' invertible factors through α. Dually, if

$$0 \to M \xrightarrow{\beta} P \to F \to 0$$

is a flasque resolution of the first type, then any morphism $M \to P'$ with P' invertible factors through β.

Proof. See Colliot-Thélène and Sansuc 1977, Lemma 1.4. □

Less well known is that resolutions of the second type also satisfy a "versality" property.

Lemma 3.2.3. Let M be a Γ-lattice. Suppose

$$0 \to M \xrightarrow{\alpha} C \to P \to 0$$

is a coflasque resolution of the second type and

$$0 \to M \xrightarrow{\gamma} N \to Q \to 0$$

is an extension of Γ-lattices with Q invertible. Then there is a morphism $\phi : N \to C$ such that $\phi \circ \gamma = \alpha$.

Let M be a Γ-lattice. Suppose

$$0 \to P \to F \xrightarrow{\alpha} M \to 0$$

is a flasque resolution of the second type and

$$0 \to Q \to N \xrightarrow{\gamma} M \to 0$$

is an extension of Γ-lattices with Q invertible. Then there is a morphism $\phi : F \to N$ such that $\gamma \circ \phi = \alpha$.

Proof. From Lemma 3.2.1, an equivalent condition that C is coflasque is that $\mathrm{Ext}^1_\Gamma(Q, C) = 0$ for all invertible modules Q. Thus from the exact sequence

$$\mathrm{Hom}_\Gamma(Q, P) \to \mathrm{Ext}^1_\Gamma(Q, M) \to \mathrm{Ext}^1_\Gamma(Q, C) = 0$$

there exists some map $\beta : Q \to P$ such that the extension

$$0 \to M \to Q \oplus_P C \to Q \to 0$$

is isomorphic to

$$0 \to M \to N \to Q \to 0.$$

The desired homomorphism $\phi : N \to C$ is the composition

$$N \cong Q \oplus_P C \to C.$$

The result for flasque resolutions follows by duality. $\square$

3.2.2 Preliminaries on algebraic tori

Definition 3.2.3. A *k-torus* is an algebraic group T over k such that $T_{\overline{k}} \cong \mathbb{G}^n_{m,\overline{k}}$ for some non-negative integer n. A torus is *split* if $T \cong \mathbb{G}^n_{m,k}$. A field extension L/k satisfying $T_L \cong \mathbb{G}^n_{m,L}$ is called a *splitting field* of the torus T. Any torus admits a finite Galois splitting field.

Recall that there is an anti-equivalence of categories between Γ_k-lattices and k-tori, which we will call *Cartier duality* (see, e.g., Voskresenskiĭ 1998). Given a torus T, the Cartier dual (or *character lattice*) $\widehat{T}$ is the Γ-lattice $\mathrm{Hom}_{\bar{k}}(\overline{T}, \mathbb{G}_{m,\bar{k}})$. Given a Γ_k-lattice M, we use $\mathscr{D}(M)$ to denote the Cartier dual torus.

Definition 3.2.4. Let T be a torus with corresponding Γ_k-lattice $M := \widehat{T}$.

1. T is *quasi-trivial* if M is permutation.

2. T is *special* if M is invertible.

3. T is *flasque* if M is flasque.

4. T is *coflasque* if M is coflasque.

Similarly, we may define flasque/coflasque resolutions of both types via Cartier duality.

As in the introduction, a *separable algebra* A over k is a finite direct sum of finite-dimensional matrix algebras over finite-dimensional division k-algebras whose centers are separable field extensions over k. Given a separable algebra A over k, we recall that $\mathrm{GL}_1(A)$ is the group scheme of units of A, i.e.,

$$\mathrm{GL}_1(A)(R) := (A \otimes_k R)^\times$$

for any commutative k-algebra R.

An étale algebra over k of degree n is a commutative separable algebra over k of dimension n. In other words, $E = F_1 \times \cdots \times F_r$ where $F_1, \ldots, F_r$ are separable field extensions of k. There is an antiequivalence between finite Γ_k-sets Ω and étale algebras E via

$$\Omega = \mathrm{Hom}_{k\text{-}\mathsf{Alg}}(E, \bar{k}) \text{ and } E = \mathrm{Hom}_{\Gamma_k\text{-}\mathsf{Set}}(\Omega, \bar{k})$$

with the natural Γ_k-action and k-algebra structure on $\bar{k}$ (see, e.g., Knus et al. 1998, §18).

Proposition 3.2.1. Let $E = F_1 \times \cdots \times F_r$ be an étale algebra over k of degree n, where $F_1, \ldots, F_r$ are separable field extensions of k. Let $T = R_{E/k}\mathbb{G}_m$ be the Weil restriction and let $\Omega := \mathrm{Hom}_k(E, \bar{k})$ be the corresponding Γ-set.

1. $T(k) = E^\times$.

2. $\widehat{T}$ is a permutation Γ_k-lattice with a canonical basis isomorphic to Ω.

3. $H^1(k, T) = 1$.

4. $H^2(k, T) = \prod_{i=1}^r \mathrm{Br}(F_i)$.

Proof. These are usual consequences of Hilbert's Theorem 90 and Shapiro's Lemma. $\square$

Corollary 3.2.1. If M is an invertible Γ_k-module then $H^1(k, \mathscr{D}(M)) = 1$. In particular, $H^1(k, T) = 1$ for any quasi-trivial torus T. Moreover, $H^1(K, T_K) = 1$ for every field extension K/k if and only if T is a special torus.

Proof. Any quasi-trivial torus corresponds to a Weil restriction of $\mathbb{G}_m$ by an étale algebra. The first result follows from the previous lemma. The second statement follows from the classification of special tori due to Colliot-Thélène Huruguen 2016, Theorem 13. □

Let us now recall some relevant rationality properties.

Definition 3.2.5. A k-variety X is *rational* if X is birationally equivalent to $\mathbb{A}^n_k$ for some $n \geq 0$. We say X is *stably rational* if $X \times \mathbb{A}^n_k$ is birational to $\mathbb{A}^m_k$ for some $n, m \geq 0$. We say X is *retract rational* if there is a dominant rational map $f : \mathbb{A}^n_k \dashrightarrow X$ that has a rational section $s : X \dashrightarrow \mathbb{A}^n_k$ such that $f \circ s$ is the identity on X.

A complete characterization of rationality of tori is still an open problem (it is not known if all stably rational tori are rational). However, stable rationality and retract rationality of a torus is completely understood via its flasque resolutions.

Theorem 3.2.1. Let T be a k-torus and

$$1 \to F \to P \to T \to 1$$

a flasque resolution of the first type.

- T is stably rational if and only if $\widehat{F}$ is stably permutation.
- T is retract rational if and only if $\widehat{F}$ is invertible.

Proof. The first item is Voskresenskiĭ 1974, Theorem 2. The second is Saltman 1984, Theorem 3.14. □

3.2.3 Flasque and coflasque resolutions of algebraic groups

We recall how one can define flasque and coflasque resolutions for more linear algebraic groups following Colliot-Thélène 2008.

Let G be a (connected) reductive algebraic group over a field k. Note that since our main application will be understanding the first Galois cohomology set of G, in characteristic 0 the reductive hypothesis is largely harmless. Let G^{ss} be the derived subgroup of G, which is semisimple, and let G^{tor} be the quotient G/G^{ss}, which is a torus.

Definition 3.2.6. Let G be a reductive algebraic group.

- The group G is *quasi-trivial* if G^{tor} is a quasi-trivial torus and G^{ss} is simply-connected.
- The group G is *coflasque* if G^{tor} is a coflasque torus and G^{ss} is simply-connected.
- A *flasque resolution* (of the first type) of G is a short exact sequence

  $$1 \to S \to H \to G \to 1$$

  where S is a flasque torus and H is quasi-trivial.
- A *coflasque resolution* (of the second type) of G is a short exact sequence

  $$1 \to P \to C \to G \to 1$$

  where P is a quasi-trivial torus and C is coflasque.

Note that the group extensions in a flaque or coflasque resolution are automatically central since the group G is connected and the automorphism group scheme of a torus has trivial connected component.

Unlike the situation for Γ-lattices and tori, the symmetry between flasque and coflasque is now broken. In this context, we do not define flasque resolutions of the second type nor coflasque resolutions of the first type. However, the flasque and coflasque resolutions we defined above always exist.

Theorem 3.2.2 (Colliot-Thélène)**.** Let G be a reductive algebraic group over k. Then there exists both a flasque resolution and coflasque resolution of G. Moreover, for any two coflasque resolutions

$$1 \to P_1 \to C_1 \to G \to 1$$
$$1 \to P_2 \to C_2 \to G \to 1$$

there is an isomorphism

$$P_1 \times C_2 \cong P_2 \times C_1.$$

Proof. The existence statements are Colliot-Thélène 2008, Proposition 3.1 and Colliot-Thélène 2008, Proposition 4.1. The isomorphism is Colliot-Thélène 2008, Proposition 4.2(i). □

Proposition 3.2.2. Suppose G is a reductive algebraic group and consider a coflasque resolution

$$1 \to P \to C \to G \to 1$$

where P is a quasi-trivial torus and C is coflasque. Suppose there exists an extension

$$1 \to S \to H \to G \to 1$$

where S is a central special torus. Then there exists a morphism $C \to H$ inducing a morphism of the extensions above that is the identity on G.

Proof. This proof is a variation of that of Proposition 4.2 of Colliot-Thélène 2008.

Let E be the fiber product of H and C over G. We have a commutative diagram

$$\begin{tikzcd}
& & 1 \arrow[d] & 1 \arrow[d] & \\
& & P \arrow[r, equal] \arrow[d] & P \arrow[d] & \\
1 \arrow[r] & S \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
& & 1 & 1 &
\end{tikzcd}$$

with exact rows and columns. From Colliot-Thélène 2008, Proposition 1.10 and 2.6, we know $H^1(C, Q) = 0$ for C coflasque and Q a quasi-trivial torus. Since S is special there is a factorization $S \to Q \to S$ of the identity for some quasi-trivial torus Q, and thus $H^1(C, S) = 0$. Arguing as in the proof of Colliot-Thélène 2008, Proposition 3.2 (or using Theorem 3.3.1 below), we conclude the group extension

$$1 \to S \to E \to C \to 1$$

is split. The composite morphism $C \to E \to H$ gives the desired result. $\square$

Proposition 3.2.3. Given a reductive algebraic group G and a coflasque resolution

$$1 \to P \to C \to G \to 1 \ ,$$

the natural morphism

$$H^1(k, C) \to H^1(k, G)$$

is injective and its image is independent of the choice of coflasque resolution.

Proof. Since P is central, the fibers of the natural morphism $H^1(k, C) \to H^1(k, G)$ are either empty or are torsors under $H^1(k, P)$. Since P is quasi-trivial, $H^1(k, P)$ is trivial and we conclude that $H^1(k, C) \to H^1(k, G)$ is injective.

Suppose

$$1 \to P' \to C' \to G \to 1 \ ,$$

is another coflasque resolution of G. From Colliot-Thélène 2008, Proposition 4.2(i) and its proof, there is an isomorphism $\alpha : P \times C' \cong P' \times C$. such that the diagram

$$\begin{tikzcd}
P \times C' \arrow[r] \arrow[d, "\alpha"] & C' \arrow[r] & G \arrow[d, equal] \\
P' \times C \arrow[r] & C \arrow[r] & G
\end{tikzcd}$$

commutes. As above, since P is quasi-trivial, the projection $P \times C' \to C'$ induces an isomorphism $H^1(k, P \times C') \cong H^1(k, C')$. Thus the composite

$$H^1(k, C') \to H^1(k, P \times C') \xrightarrow{\alpha} H^1(k, P' \times C) \to H^1(k, C)$$

is an isomorphism and induces equality of the images in $H^1(k, G)$. $\square$

Note that a flasque resolution of a reductive algebraic group G does not in general give rise to a flasque resolution of its abelianization. However, this does occur if G is coflasque:

Proposition 3.2.4. If C is a coflasque reductive algebraic group, then any flasque resolution of the first type gives rise to a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S \arrow[r] & H^{tor} \arrow[r] & C^{tor} \arrow[r] & 1
\end{tikzcd}$$

with exact rows, where H is a quasi-trivial algebraic group, S is a flasque torus, and the vertical maps are abelianizations. Note that both rows are flasque resolutions.

Proof. The only potential problem is that abelianization is not left-exact in general. The morphism $\varphi : H \to C$ induces a surjective morphism $H' \to C'$ of their derived subgroups with commutative kernel $H' \cap S$. However, since C is coflasque, the semisimple algebraic group C' is simply-connected by definition. Thus $H' \to C'$ is an isomorphism and $H' \cap S = 1$. Consider the map $S \to \varphi^{-1}(C')/H'$. The kernel is $S \cap H' = 1$. Given $h \in \varphi^{-1}(C')$, since $\varphi|_{H'}$ is an isomorphism, there is some h' with $\varphi(hh') = 1$ so $S \to \varphi^{-1}(C')/H'$ is surjective. We conclude that

$$1 \to S \to H/H' \to C/C' \to 1$$

is exact. $\square$

3.3 Cohomological Invariants

We review the notion of a *cohomological invariant* following Serre 2003. Fix a base field k and recall our notation k-Fld for the category of field extensions of k. We consider two functors

$$A : k\text{-}\mathsf{Fld} \to \mathsf{Sets}_*$$

and

$$H : k\text{-}\mathsf{Fld} \to \mathsf{Ab}\ .$$

A *normalized H-invariant of A* is a morphism of functors $A \to H$. The group of all such invariants will be denoted $\mathrm{Inv}_*(A, H)$.

Remark 3.3.1. We demand a priori that A is a functor into *pointed* sets. This explains the adjective "normalized." This condition is harmless as a general H-invariant of A can be written uniquely as the sum of a normalized invariant and a "constant" invariant coming from $H(k)$.

The two kinds of functors we will consider are as follows. Given an algebraic group G over k, we may view Galois cohomology

$$H^i(-, G) : k\text{-}\mathsf{Fld} \to \mathsf{Sets}_*$$

as a functor (the codomain may be interpreted as Grp if $i = 0$ or Ab if G is commutative). If the functor A is $H^1(-, G)$ and the functor H is $H^i(-, C)$ for G, C algebraic groups, we let

$$\operatorname{Inv}^i_*(G, C)$$

denote the group of normalized H-invariants of A.

Let S be a torus. Recall that $\operatorname{Ext}^1_k(G, S)$ is the group of central extensions of algebraic groups

$$1 \to S \to H \to G \to 1 \tag{3.3.1}$$

up to equivalence under the usual Baer sum. At the risk of some ambiguity, we will use $[H]$ to denote the class of such an extension. Given such an extension, there is a connecting homomorphism

$$\partial_H : H^1(k, G) \to H^2(k, S)$$

from the long exact sequence in Galois cohomology.

We define a pairing

$$\beta : \operatorname{Ext}^1_k(G, S) \times H^1(k, G) \to H^2(k, S) \tag{3.3.2}$$

via $\beta([H], [X]) := \partial_H([X])$. Fixing a G-torsor $X \to \operatorname{Spec}(k)$ representing a class in $H^1(k, G)$ we obtain a function

$$\delta_X : \operatorname{Ext}^1_k(G, S) \to H^2(k, S)$$

defined by $\delta_X([H]) = \partial_H([X])$.

Lemma 3.3.1. The pairing β is additive in the first variable. In other words, for all G-torsors $X \to \operatorname{Spec}(k)$, the function δ_X is a group homomorphism.

Proof. Suppose H and H' are two extensions in $\operatorname{Ext}^1_k(G, S)$. Let H'' denote the Baer sum of H and H'. Recall that this means there is an algebraic group K and a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & S \times S \arrow[r] \arrow[d, equal] & H \times H' \arrow[r] & G \times G \arrow[r] & 1 \\
1 \arrow[r] & S \times S \arrow[r] \arrow[d, "\mu"] & K \arrow[r] \arrow[u] \arrow[d] & G \arrow[r] \arrow[u, "\Delta"'] \arrow[d, equal] & 1 \\
1 \arrow[r] & S \arrow[r] & H'' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

where $\Delta : G \to G \times G$ is the diagonal map and $\mu : S \times S \to S$ is the multiplication.

Now we compute that

$$\begin{aligned}
&\delta_X([H]) + \delta_X([H'])\\
=&\mu_* \left(\delta_X([H]), \delta_X([H'])\right)\\
=&\mu_* \left(\partial_H([X]), \partial_{H'}([X])\right)\\
=&\mu_* \left(\partial_{H\times H'}([X \times X])\right)\\
=&\mu_* \left(\partial_K([X])\right)\\
=&\partial_{H''}([X]) = \delta_X([H'']) = \delta_X([H] + [H'])
\end{aligned}$$

where $\mu_* : H^2(k, S^2) \to H^2(k, S)$ is induced from multiplication and each ∂ is the connecting homomorphism from H^1 to H^2 for each exact sequence in the commutative diagram above. Thus δ_X is a homomorphism as desired. $\square$

Given a central extension H, we obtain a function $\beta(-, [H])$ from $H^1(k, G) \to H^2(k, S)$ that is functorial in k. Thus there is a canonical group homomorphism

$$\mathrm{Ext}^1_k(G, S) \to \mathrm{Inv}^2_*(G, S) \ . \tag{3.3.3}$$

The main goal of this section is to prove Theorem 2; that is, (3.3.3) is an isomorphism when S is a special torus.

In Blinstein and Merkurjev 2013, Theorem 2.4, it is show that

$$\mathrm{Pic}(G) \cong \mathrm{Inv}^2_*(G, \mathbb{G}_m) \ .$$

Moreover, it is known that $\mathrm{Pic}(G) \cong \mathrm{Ext}^1_k(G, \mathbb{G}_m)$. Thus, using a Weil restriction argument, one can possibly establish that the two groups in 3.3.3 are isomorphic.

However, in the proof of Blinstein and Merkurjev's Theorem, the specific isomorphism constructed is somewhat mysterious. Their map comes from an exact sequence in Sansuc 1981, Proposition 6.10, but it is not clear that is the same as the map given in Equation (3.3.3). That these two maps are similar has been noticed before (see Colliot-Thélène 2008, Remarque after Proposition 8.2), but it is not clear they are equal. Thus Theorem 2 appears to be new even in the case where $S = \mathbb{G}_m$.

3.3.1 Torsors over torsors

We recall Corollary 5.7 from Colliot-Thélène 2008:

Theorem 3.3.1 (Colliot-Thélène)**.** Let G be a connected algebraic group and S an algebraic group of multiplicative type. There is an exact sequence

$$1 \to \mathrm{Ext}^1_k(G, S) \xrightarrow{\psi} H^1(G, S) \xrightarrow{e^*} H^1(k, S)$$

where $e : \mathrm{Spec}(k) \to G$ is the inclusion of the identity and ψ is the "forgetful map" that takes the class of a central extension

$$1 \to S \to H \to G \to 1$$

to the class of the S-torsor $H \to G$.

In particular, if S is special, then ψ is an isomorphism. Thus, if $H \to G$ is an S-torsor, then H has a unique structure of an algebraic group compatible with G and S. We will generalize this and see that an S-torsor over a G-torsor is itself a torsor (for an appropriate algebraic group).

Lemma 3.3.2. Let X and Y be smooth varieties over k, with Y separably rational and $Y(k) \neq \emptyset$. Let S be a special torus. Then the canonical homomorphism

$$H^1(X, S) \times H^1(Y, S) \to H^1(X \times Y, S)$$

is an isomorphism.

Proof. When $S = \mathbb{G}_m$, recall that $\operatorname{Pic}(X) = H^1(X, S)$. Thus the case of $S = \mathbb{G}_m$ is exactly Sansuc 1981, Lemme 6.6, which states that the canonical map

$$\operatorname{Pic}(X) \times \operatorname{Pic}(Y) \to \operatorname{Pic}(X \times Y)$$

is an isomorphism under the same conditions on X and Y. For a finite separable field extension L/k, we have a canonical isomorphism $H^1(X, S) \cong H^1(X_L, \mathbb{G}_m)$ where $S = R_{L/k}\mathbb{G}_m$ is the Weil restriction. Thus, the lemma holds when S is a Weil restriction. Since there is a canonical isomorphism

$$H^1(X, S \times S') \cong H^1(X, S) \times H^1(X, S')$$

for tori S and S', the lemma holds for quasi-trivial tori (since they are precisely the products of Weil restrictions). Recall that special tori correspond to invertible Γ_k-modules, which are direct summands of permutation Γ_k-modules. Thus for any special torus S, there exists a quasi-trivial torus of the form $S \times S'$. Since the composite $S \to S \times S' \to S$ is the identity, functoriality of H^1 shows that the result holds for all special tori. $\square$

Remark 3.3.2. The hypothesis that S is special is crucial. For example, the lemma is false when $k = \mathbb{R}$, $X = \operatorname{Spec}\mathbb{C}$, $Y = \operatorname{Spec}\mathbb{R}$, and S is the non-split real one-dimensional torus.

We will also require the following amplification of Rosenlicht's Lemma:

Lemma 3.3.3. If X and Y are varieties over k and S is a special torus, then the canonical homorphism $\mathrm{Hom}_k(X,S) \times \mathrm{Hom}_k(Y,S) \to \mathrm{Hom}_k(X \times_k Y, S)$ is an isomorphism.

Proof. Recall that $\mathrm{Hom}_k(X,\mathbb{G}_m) = k[X]^\times/k^\times$ is just the group of invertible regular functions on X. Thus the result for $S = \mathbb{G}_m$ is simply Rosenlicht's Lemma Colliot-Thélène and Sansuc 1977, Lemme 10. For a finite separable field extension L/k, we have $\mathrm{Hom}_k(X, R_{L/k}\mathbb{G}_m) = \mathrm{Hom}_L(X_L, \mathbb{G}_m)$. Thus the result holds for Weil restrictions $S = R_{L/k}\mathbb{G}_m$. Since $\mathrm{Hom}(X,-)$ is additive in the second variable, it applies to products of Weil restrictions; namely, all quasi-trivial tori Q. Since special tori S possess factorizations $S \to Q \to S$ of the identity for some quasi-trivial torus, the result holds for all special tori. $\square$

Theorem 3.3.2. Let G be a reductive algebraic group and suppose S is a special torus. Let $X \to \mathrm{Spec}(k)$ be a G-torsor and $Y \to X$ be a S-torsor. Then there exists a central extension

$$1 \to S \to H \to G \to 1,$$

unique up to isomorphism of extensions, along with an H-action on Y such that:

1. the composite $Y \to X \to \mathrm{Spec}(k)$ is an H-torsor,
2. restriction of the H-action yields the existing S-action on Y, and
3. the induced H-action on the quotient X factors through the map $H \to G$.

Proof. It is well known that every reductive group is rational over a separably closed field. In characteristic 0, this is due to Chevalley 1954 — in this case, for arbitrary linear algebraic groups. However, we could not find a direct reference for the case of positive characteristic so we sketch a proof that G is separably-rational here. From

Gille and Polo 2011, Exp. XXII Corollary 2.4, there is a finite separable extension L/k such that G_L has a split maximal torus T and a system of roots. A set of positive roots and negatives roots determines opposite Borels B, B' Gille and Polo 2011, Exp. XXII Proposition 5.5.1 and 5.9.2. The unipotent radical B^u of B is isomorphic to an affine space, and thus $B' \cong B'^u \rtimes T$ is rational as well. The natural map

$$B^u \times B' \to G_L$$

is an open immersion Gille and Polo 2011, Exp. XXII Proposition 5.9.3. Hence, G_L is L-rational.

Since G is separably rational with a rational point, we can apply Lemma 3.3.2 to get a canonical isomorphism

$$\gamma : H^1(X,S) \oplus H^1(G,S) \to H^1(X \times G, S)$$

given by

$$\gamma(\alpha,\beta) = \mathrm{pr}_1^*(\alpha) + \mathrm{pr}_2^*(\beta)$$

where $\mathrm{pr}_1 : X \times G \to X$ and $\mathrm{pr}_2 : X \times G \to G$ are the projection maps. We interpret γ geometrically. Let $Y \to X$ and $H \to G$ be S-torsors represented by the classes α and β as above. Then $\mathrm{pr}_1^*(\alpha)$ represents the S-torsor $Y \times G \to X \times G$ and $\mathrm{pr}_2^*(\beta)$ represents $X \times H \to X \times G$. Their sum $\gamma(\alpha,\beta)$ is the quotient of $(Y \times G) \times_{(X \times G)} (X \times H) \cong Y \times H$ by the diagonal S action.

Let $\pi_X : H^1(X \times G, S) \to H^1(X,S)$ and $\pi_G : H^1(X \times G, S) \to H^1(G,S)$ be the projections obtained from the inverse of γ. Let $\sigma : X \times G \to X$ be the action morphism. We define

$$\varphi : H^1(X,S) \to H^1(G,S)$$

as the composition $\pi_G \circ \sigma^*$.

Let $Y \to X$ be an S-torsor. Then $Z = \sigma^* Y$ is an S-torsor over $X \times G$. Since γ is an isomorphism, there exists an S-torsor $W \to X$ and an S-torsor $H \to G$ (both unique up to isomorphism) such that the $(S \times S)$-torsor $\tau : W \times H \to X \times G$ factors through Z via the diagonal S-action quotient. In particular, $\varphi([X]) = [H]$. Let $\iota : X \to X \times G$ be the inclusion via $X \times e_G$. Since $\sigma \circ \iota$ and $\mathrm{pr}_1 \circ \iota$ are the identity on X, we conclude that $\iota^*([Z]) = [Y]$ and so $W \cong Y$.

Now, we endow H with an algebraic group structure sitting in a central extension

$$1 \to S \to H \to G \to 1$$

via Theorem 3.3.1. We have a commutative diagram

$$\begin{tikzcd} Y \times H \arrow[r, "\tau"] \arrow[d] & Y \arrow[d] \\ X \times G \arrow[r, "\sigma"] & X \end{tikzcd}$$

such that $\tau(ys, ht) = st\tau(y, h)$ for $y \in Y(\bar{k})$, $h \in H(\bar{k})$, and $s, t \in S(\bar{k})$.

Note that τ is not canonical and may not necessarily be an action map for H. However, let $\chi : Y \to Y$ be the composition $Y \times \{e\} \to Y \times H \to Y$. We replace τ with $\chi^{-1} \circ \tau$ and claim that now τ is a group action.

To check that τ is a group action, it suffices to assume k is algebraically closed. By the modification above, we have $\tau(y, e_H) = y$ for $y \in Y$.

Now $\sigma : X \times G \to X$ is a right action and thus there exists a homomorphism $\omega : Y \times H \times H \to S$ factoring through $X \times G \times G$ such that

$$\tau(y, h_1 h_2) = \tau(\tau(y, h_1), h_2)\omega(y, h_1, h_2) \tag{3.3.4}$$

for $y \in Y$ and $h_1, h_2 \in H$. By Lemma 3.3.3,

$$\omega(y, h_1, h_2) = \chi_1(y)\chi_2(h_1)\chi_3(h_2)$$

where $\chi_1 : Y \to S$ is a map factoring through X and $\chi_2, \chi_3 : H \to S$ are morphisms factoring through G.

Taking $h_1 = h_2 = e_H$ in (3.3.4), we find

$$y = y\chi_1(y)\chi_2(e_H)\chi_3(e_H),$$

which shows that χ_1 is a constant function. Taking $h_1 = e_H$ in (3.3.4), we find

$$\tau(y, h) = \tau(y, h)\chi_1(y)\chi_2(e_H)\chi_3(h),$$

which shows that χ_3 is a constant function. Similarly, taking $h_2 = e_H$ shows that χ_2 is a constant function. Thus ω is a constant function. Since $\omega(y, e_H, e_H) = e_S$ and ω is constant, we conclude that τ is an action. $\square$

3.3.2 An exact sequence of Sansuc

Given a G-torsor $\pi : X \to Y$, a long exact sequence is constructed by Sansuc 1981, Proposition 6.10, which contains the important subsequence:

$$\mathrm{Pic}(X) \xrightarrow{\varphi} \mathrm{Pic}(G) \to \mathrm{Br}(Y) \xrightarrow{\pi^*} \mathrm{Br}(X).$$

However, the map $\mathrm{Pic}(G) \to \mathrm{Br}(Y)$ is constructed by a series of maps obtained from spectral sequences and thus is somewhat obscure. For our applications, we need to know a specific interpretation for this map.

The following theorem can be seen as a variation of Sansuc's result, which extends $\mathbb{G}_m$ to a general special torus S and explicitly describes the maps occurring in the sequence.

Theorem 3.3.3. Let G be a reductive algebraic group. Let S be a special torus and suppose $\pi : X \to \mathrm{Spec}(k)$ is a G-torsor. Then the sequence

$$H^1(X, S) \xrightarrow{\varphi} H^1(G, S) \xrightarrow{\delta_X} H^2(k, S) \xrightarrow{\pi^*} H^2(X, S) \tag{3.3.5}$$

is exact at $H^1(G, S)$ and $\pi^* \circ \delta_X$ is trivial.

Proof. We show first the following claim.

Claim: The composite $H^1(X,S) \xrightarrow{\varphi} H^1(G,S) \xrightarrow{\delta_X} H^2(k,S)$ is trivial.

Let $Y \to X$ be an S-torsor. By Theorem 3.3.1, there is an algebraic group H representing $\varphi([Y])$ in $H^1(G,S)$. By construction, $Y \to \mathrm{Spec}(k)$ is an H-torsor whose image under $H^1(k,H) \to H^1(k,G)$ is the isomorphism class of X. In particular, $\partial_H([X]) = 0$. We have $\delta_X([H]) = \partial_H([X])$, so the claim follows.

Claim: $\mathrm{Im}(\varphi) = \ker(\delta_X)$.

Suppose H is a central extension of G by S such that $\delta_X([H]) = 0$. Then $\partial_H([X]) = 0$. This implies there exists a an H-torsor Y and a $(H \to G)$-equivariant map $Y \to X$. Thus $Y \to X$ is an S-torsor and we conclude $\varphi([Y]) = [H]$.

Claim: The composite $H^1(G,S) \xrightarrow{\delta_X} H^2(k,S) \xrightarrow{\pi^*} H^2(X,S)$ is trivial.

Let H be a group extension representing an element in $H^1(G,S)$. Recall that $\partial_H([X])$ can be interpreted as (the equivalence class of) the S-gerbe $\mathcal{G}$ of lifts of X to H. Specifically, $\mathcal{G}(U)$ is the category of H-torsors $T \to U$ with $(H \to G)$-equivariant maps $T \to X_U$. To prove the lemma, we want to show the pullback of $\mathcal{G}$ along $X \to \mathrm{Spec}(k)$ is a trivial gerbe. This is equivalent to showing that $\mathcal{G}(X)$ is non-empty. Thus, we need to find an H-torsor $T \to X$ with an $(H \to G)$-equivariant map $T \to X \times X$. Of course, there is an isomorphism $X \times G \cong X \times X$ by the definition of a G-torsor, so the composite $X \times H \to X \times G \cong X \times X$ is the desired map. $\square$

3.3.3 Blinstein and Merkurjev

Lemma 3.3.4. Let X be a regular variety over k and let S be a special torus over k. Then the homomorphism obtained by the pullback of the generic point

$$H^2(X,S) \to H^2(k(X),S)$$

is injective.

Proof. The case of $S = \mathbb{G}_m$ is simply that case of Brauer groups $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$, which is standard (see, e.g., Milne 1980, Example 2.22). Thus, for any étale k-algebra L, we have the injectivity of $H^2(X_L, \mathbb{G}_m) \to H^2(L \otimes_k k(X), \mathbb{G}_m)$ and thus injectivity of $H^2(X, Q) \to H^2(k(X), Q)$ for all quasi-trivial tori $Q = R_{L/k}\mathbb{G}_m$. For a special torus S, the result follows from the factorization $S \to Q \to S$ of the identity through a quasi-trivial torus. □

The following is a mild generalization of Blinstein and Merkurjev 2013, Lemma 2.3.

Lemma 3.3.5. Let G be an algebraic group and let S be a special torus over k, and suppose K/k is a field extension such that k is algebraically closed in K. Then the natural map $H^1(G, S) \to H^1(G_K, S_K)$ is an isomorphism.

Proof. When $S = \mathbb{G}_m$, this result is Blinstein and Merkurjev 2013, Lemma 2.3, which incidentally uses both a coflasque resolution of G and a piece of Sansuc 1981, Proposition 6.10 in the proof. Thus we may assume the lemma holds for $S = \mathbb{G}_m$.

Consider F/k a finite separable extension and let $S = R_{F/k}\mathbb{G}_m$. Note that $FK = F \otimes_k K$ is a field since k is algebraically closed in K. From the $\mathbb{G}_m$ case of the lemma, the natural map

$$H^1(G_F, \mathbb{G}_{m,F}) \to H^1(G_{FK}, \mathbb{G}_{m,FK})$$

is isomorphism. By the Weil restriction adjunction we see that

$$H^1(G, R_{F/k}\mathbb{G}_{m,F}) \to H^1(G_K, R_{FK/K}\mathbb{G}_{m,FK})$$

is an isomorphism. Note that $R_{FK/K}\mathbb{G}_{m,FK}$ is canonically isomorphic to $(R_{F/k}\mathbb{G}_{m,F})_K$ since they both represent the functor

$$A \mapsto ((F \otimes_k K) \otimes_K A)^\times$$

on K-algebras A. Thus, the natural map $H^1(G, S) \to H^1(G_K, S_K)$ is an isomorphism in the case where $S = R_{F/k}\mathbb{G}_m$.

Since the functor $H^1(G, -)$ preserves finite limits, the result holds when S is quasi-trivial. If S is a special torus, then there is a quasi-trivial torus Q along with morphisms $S \to Q \to S$ that compose to the identity. Thus the result now follows by functoriality of $H^1(G, -)$. $\square$

Proof of Theorem 2. The following is adapted from the proof of Blinstein and Merkurjev 2013, Theorem 2.4. Recall that we want to show that the map

$$\mathrm{Ext}^1_k(G, S) \to \mathrm{Inv}^2_*(G, S),$$

which takes an extension ξ to its connecting homomorphism δ_ξ, is a group isomorphism. Precomposing with the canonical identification $H^1(G, S) \cong \mathrm{Ext}^1_k(G, S)$ we obtain

$$\nu : H^1(G, S) \to \mathrm{Inv}^2_*(G, S),$$

which we will prove is an isomorphism.

The remainder of the proof makes use of *versal torsors* — see Serre 2003, Section 5. Since G is a linear algebraic group, there exists an embedding of algebraic groups $G \to \mathrm{GL}_n$ for some n. The quotient $\mathrm{GL}_n \to \mathrm{GL}_n / G$ is a G-torsor and the pullback by the generic point $\pi \colon T \to \mathrm{Spec}(K)$ is a versal G-torsor. Consider the map

$$\theta : \mathrm{Inv}^2_*(G, S) \to H^2(K, S)$$

that sends a cohomological invariant α to its value $\alpha(T)$ for the versal torsor $T \to \mathrm{Spec}(K)$. By Serre 2003, Theorem 12.3, the map θ is injective.

We claim $H^1(T, S) = 0$. From Colliot-Thélène 2008, Définition 1.8, we recall that a geometrically-integral variety X over k is *finie-factorielle* if $\mathrm{Pic}(X_K) = 0$ for all finite separable field extensions K/k. From Colliot-Thélène 2008, Proposition 1.9, if X is smooth and finie-factorielle, then so is every open subset. In particular, $\mathrm{GL}_{n,k}$ is finie-factorielle since it is an open subset of affine space. From Colliot-Thélène 2008, Proposition 1.10, $H^1(X, Q) = 0$ for every finie-factorielle X and quasi-trivial torus

Q; Thus $H^1(U, S) = 0$ for any open subset U of GL_n since S is a direct multiplicand of some such Q. From *The Stacks project*, Tag 09YQ, we conclude

$$H^1(T, S) = \mathrm{colim}_U\, H^1(U, S) = 0$$

where the limit is over all open subsets U of GL_n containing T.

An element $H \in H^1(G, S)$ can be interpreted as a group extension

$$1 \to S \to H \to G \to 1$$

and therefore $\nu(H)(K)$ is the connecting homomorphism

$$\partial_{H_K} : H^1(K, G_K) \to H^2(K, S_K);$$

in particular, $(\theta \circ \nu)(H) = \partial_{H_K}(T)$. Let $j : H^1(G, S) \to H^1(G_K, S_K)$ be the isomorphism from Lemma 3.3.5. We see that $(\delta_T \circ j)(H) = \delta_T(H_K)$, which is equal to $\partial_{H_K}(T)$ by definition of δ_T. Thus, there is a commutative diagram

$$\begin{tikzcd}
 & H^1(G,S) \arrow[r, "\nu"] \arrow[d, "j"] & \mathrm{Inv}^2_*(G,S) \arrow[d, "\theta"] & \\
H^1(T,S_K) \arrow[r] & H^1(G_K,S_K) \arrow[r, "\delta_T"] & H^2(K,S_K) \arrow[r, "\pi^*"] & H^2(T,S_K)
\end{tikzcd}$$

where the bottom sequence is (3.3.5) for the G_K-torsor T. Since $H^1(T, S_K) = 0$ and j is an isomorphism, we see that δ_T is injective.

The pullback map $i : H^2(T, S) \to H^2(k(T), S)$ is injective by Lemma 3.3.4. The composite

$$i \circ \pi^* \circ \theta \colon\ \mathrm{Inv}^2_*(G, S) \to H^2(k(T), S)$$

takes a cohomological invariant $\alpha \in \mathrm{Inv}^2_*(G, S)$ to $\alpha\,(T \times_K \mathrm{Spec}(k(T)))$ since α is a natural transformation from fields to sets. Note that the generic point lifts to a rational point of the torsor $T_{\mathrm{Spec}(k(T))}$. Thus, the torsor T is trivialized by $\mathrm{Spec}(k(T))$ and we conclude that $\mathrm{Im}(\theta) \subseteq \ker(\pi^*)$. It follows that ν is an isomorphism. $\square$

3.4 Connecting Coflasque Resolutions and Cohomological Invariants

The purpose of this section is to prove Theorem 1. We will actually prove a stronger theorem:

Theorem 3.4.1. Let G be a reductive algebraic group G defined over a field k. The following sets are equal:

1. $\maltese(k, G) = \bigcap_A \ker\left(H^1(k, G) \to H^1(k, \mathrm{Aut}(A))\right)$ where the intersection runs over all separable k-algebras A with a G-action.

2. $\maltese_{Br}(k, G) := \bigcap_E \bigcap_\alpha \ker\left(\alpha(k)\colon H^1(k, G) \to \mathrm{Br}(E)\right)$ where the intersections run over all étale algebras E and all normalized cohomological invariants α.

3. $\maltese_{qt}(k, G) := \bigcap_S \bigcap_\alpha \ker\left(\alpha(k)\colon H^1(k, G) \to H^2(k, S)\right)$ where the intersections run over all quasi-trivial tori S and all normalized cohomological invariants α.

4. $\maltese_{sp}(k, G) := \bigcap_S \bigcap_\alpha \ker\left(\alpha(k)\colon H^1(k, G) \to H^2(k, S)\right)$ where the intersections run over all special tori S and all normalized cohomological invariants α.

5. $\mathrm{Im}\left(H^1(k, C) \to H^1(k, G)\right)$ where $1 \to P \to C \to G \to 1$ is a coflasque resolution of the second type.

Moreover, it suffices to consider only one element in each intersection.

We begin with the following lemma:

Lemma 3.4.1. Let G be a reductive algebraic group G defined over a field k. Let

$$1 \to P \to C \to G \to 1$$

be a coflasque resolution of the second type. For any special torus S and any normalized cohomological invariant $\alpha \in \mathrm{Inv}^2_*(G, S)$, there exists a group homomorphism $f\colon P \to S$ such that α is equal to the composite

$$H^1(k, G) \xrightarrow{\partial_C} H^2(k, P) \xrightarrow{f_*} H^2(k, S)$$

and $\ker(\alpha)$ contains the image of $H^1(k, C) \to H^1(k, G)$.

Proof. By Theorem 2, every α is obtained as a connecting homomorphism from some extension

$$1 \to S \to M \to G \to 1 \ .$$

By Proposition 3.2.2, there exists a homomorphism $m : P \to S$ coming from a morphism of extensions. Applying Galois cohomology, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
H^1(k, C) \arrow[r] \arrow[d] & H^1(k, G) \arrow[r] \arrow[d, equal] & H^2(k, P) \arrow[d] \\
H^1(k, M) \arrow[r] & H^1(k, G) \arrow[r, "\alpha(k)"] & H^2(k, S)
\end{tikzcd}$$

Thus $H^1(k, C)$ is in the kernel of α as desired. $\square$

From §23 of Knus et al. 1998, we recall some standard facts about automorphisms of separable algebras. Let A be a separable algebra over k with center $Z(A)$ (an étale algebra over k). Recall that the connected component $\mathrm{Aut}_k(A)^\circ$ of the group scheme of algebra automorphisms of $\mathrm{Aut}_k(A)$ is the kernel of the restriction map $\mathrm{Aut}_k(A) \to \mathrm{Aut}_k(Z(A))$.

We have an exact sequence

$$1 \to \mathrm{GL}_1(Z(A)) \to \mathrm{GL}_1(A) \to \mathrm{Aut}_k(A)^\circ \to 1$$

where $\mathrm{GL}_1(B)$ is the group scheme of units of a k-algebra B (this is a consequence of the Skolem-Noether theorem). We define $\mathrm{PGL}_1(A)$ as the quotient

$$\mathrm{GL}_1(A)/\,\mathrm{GL}_1(Z(A)) \cong \mathrm{Aut}_k(A)^\circ.$$

Lemma 3.4.2. Suppose there is a central extension of algebraic groups

$$1 \to S \to H \to G \to 1$$

and a homomorphism $m : S \to P$ where P is a quasi-trivial torus. Then there exists a separable algebra A such that $P \cong \mathrm{GL}_1(Z(A))$ and there is commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & S \arrow[r] \arrow[d, "m"] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & P \arrow[r] & \mathrm{GL}_1(A) \arrow[r] & \mathrm{PGL}_1(A) \arrow[r] & 1 \end{tikzcd}$$

with exact rows.

Proof. By taking the pushout of H along $S \to P$, we may assume that $S = P$ and the morphism $S \to P$ is the identity. We begin by proving the theorem in the case where $S = \mathbb{G}_m$. Let $\rho : H \to \mathrm{GL}(V)$ be a faithful algebraic representation of H where V is a k-vector space. Recall that tori are linearly reductive over any field, so the restricted representation $\rho|_S$ has a canonical decomposition $V = V_1 \oplus \cdots \oplus V_n$ into isotypic components where $\rho|_S$ acts on V_i via a direct sum of many copies of a single irreducible representation $\sigma_i : S \to \mathbb{G}_m$. Observe that H cannot permute these components since S is central, thus each V_i is H-stable. Since the representation ρ is faithful and S is central, at least one σ_i must be a faithful representation of S. Since $S = \mathbb{G}_m$, either σ_i is the identity or the inversion. In the latter case, $\sigma_i^\vee$ is the identity. Thus there exists a representation of H on V_i which restricts to scalar multiplication on $S = \mathbb{G}_m$. Thus, the theorem follows when $S = P = \mathbb{G}_m$ if we set $A = \mathrm{End}(V_i)$.

We now consider the general case where $S = P$ is quasi-trivial. It suffices to assume that $P = R_{K/k}\mathbb{G}_m$ for a finite separable field extension K/k of degree n. Indeed, quasi-trivial tori are products of such tori; so the general result follows by taking the product of the constructions.

Let $\pi : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ be the morphism corresponding to the field extension K/k. For brevity and clarity we will write $L(X) = X_K$ for k-varieties X and $R(Y) =$

$R_{K/k}(Y)$ for K-varieties Y, which emphasizes that scalar extension, L, is a left adjoint to Weil restriction R. We have an adjoint pair, so we denote the counit by $\epsilon : LR \to \mathrm{id}$ and the unit by $\eta : \mathrm{id} \to RL$.

Let $f : RL(\mathbb{G}_m) = R_{K/k}(\mathbb{G}_{m,K}) \to H$ be the inclusion of S into H. Define the K-group J as the pushout

$$\begin{tikzcd} LRL(\mathbb{G}_m) \arrow[r, "\epsilon_{L\mathbb{G}_m}"] \arrow[d, "Lf"] & L(\mathbb{G}_m) \arrow[d, "g"] \\ L(H) \arrow[r, "h"] & J \end{tikzcd}$$

with g and h the canonical maps. Since the lemma has been proven for the case $S = \mathbb{G}_m$, we have an embedding $\rho : J \to \mathrm{GL}_{n,K}$ for some n such that $\rho \circ g$ is the identity on scalar matrices.

We have the following commutative diagram:

$$\begin{tikzcd} RL(\mathbb{G}_m) \arrow[r, "\eta_{RL\mathbb{G}_m}"] \arrow[d, "f"] & RLRL(\mathbb{G}_m) \arrow[r, "R\epsilon_{L\mathbb{G}_m}"] \arrow[d, "RLf"] & RL(\mathbb{G}_m) \arrow[d, "Rg"] \\ H \arrow[r, "\eta_H"] & RL(H) \arrow[r, "Rh"] & R(J) \end{tikzcd}$$

where the left square commutes due to naturality of η. The top row composes to be the identity since $R\epsilon \circ \eta R = \mathrm{id}$ by standard facts regarding adjunctions.

Let A be the k-algebra of $n \times n$ matrices over K. Since $R_{K/k}(\mathrm{GL}_{n,K})$ is canonically isomorphic to $\mathrm{GL}_1(A)$, the composition

$$R(\rho \circ h) \circ \eta_H : H \to R_{K/k}(\mathrm{GL}_{n,K})$$

gives the desired map. The isomorphism $S \to Z(\mathrm{GL}_1(A))$ is given by the top row of the diagram above. $\square$

With this technical lemma in hand, we are finally able to prove Theorem 3.4.1 (and thus Theorem 1).

Proof of Theorem 3.4.1. Since $\mathrm{Br}(E) = H^2(k, R_{E/k}\mathbb{G}_m)$ for an étale k-algebra E, we conclude immediately that (b) and (c) are equal. Since a quasi-trivial torus is, in particular, special, the equality of (c), (d) and (e) follow from Lemma 3.4.1.

Thus, the theorem is proven provided we can show $\text{Ж}(k, G) = \text{Ж}_{qt}(k, G)$.

Suppose $x \in \text{Ж}_{qt}(k, G)$. Let A be an algebra with a group action $\alpha : G \to \mathrm{Aut}(A)$. Since G is connected, we may assume $\alpha : G \to \mathrm{Aut}(A)^\circ$ instead. We have a composition

$$\beta\colon H^1(k, G) \xrightarrow{\partial_\alpha} H^1(k, \mathrm{Aut}(A)^\circ) \hookrightarrow H^2(k, \mathrm{GL}_1(Z(A)))$$

where the second arrow is injective by Hilbert 90. In particular, this composition gives rise to a cohomological invariant and thus $\beta(x) = 0$ since $x \in \text{Ж}_{qt}(k, G)$; thus $\partial_\alpha(x) = 0$. We conclude that $x \in \text{Ж}(k, G)$.

Suppose $x \in \text{Ж}(k, G)$. Consider a quasi-trivial torus P and a cohomological invariant $\alpha \in \mathrm{Inv}^2_*(G, P)$. From Theorem 2, the functor α is the connecting homomorphism induced from a central extension

$$1 \to P \to H \to G \to 1 \ .$$

From Lemma 3.4.2, we may construct a separable algebra A and a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & P \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P \arrow[r] & \mathrm{GL}_1(A) \arrow[r] & \mathrm{PGL}_1(A) \arrow[r] & 1
\end{tikzcd}$$

with exact rows. Applying Galois cohomology we obtain a factorization

$$H^1(-, G) \to H^1(-, \mathrm{PGL}_1(A)) \to H^2(-, P)$$

of the functor α. We conclude that $x \in \text{Ж}_{qt}(k, G)$. $\square$

3.5 Coflasque algebraic groups over particular fields

3.5.1 General statements and low cohomological dimension

An algebraic group G over k is *special* if and only if $H^1(K, G_K) = *$ for every field extension K/k (see Huruguen 2016).

Proposition 3.5.1. If G is a reductive group then $\text{Ж}(-, G)$ is trivial if and only if C is special, where

$$1 \to P \to C \to G \to 1$$

is a coflasque resolution of the second type.

Proof. By Theorem 1, we have an isomorphism of functors $\text{Ж}(-, G) \cong H^1(-, C)$; the latter is trivial if and only if C is special by definition. $\square$

Proposition 3.5.2. Let T be a torus over k. Then, $\text{Ж}(-, T)$ is a stable birational invariant of T. Moreover, $\text{Ж}(-, T)$ is trivial if and only if T is retract rational.

Proof. Let

$$1 \to P \to C \to T \to 1$$

be a coflasque resolution of T of the second type. Assume we have an exact sequence

$$1 \to Q \to E \to T \to 1$$

with Q invertible. Taking the fiber product we obtain a commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & P \arrow[r, equal] \arrow[d] & P \arrow[d] & \\
1 \arrow[r] & Q \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Q \arrow[r] & E \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

with exact rows and columns. Taking duals of the middle column we get a short exact sequence

$$1 \to \hat{C} \to \hat{H} \to \hat{Q} \to 1$$

whose associated long exact sequence includes

$$H^1(\Gamma', \hat{C}) \to H^1(\Gamma', \hat{H}) \to H^1(\Gamma', \hat{Q})$$

for any $\Gamma' \leq \Gamma$. Since the outer two terms vanish, so does the middle. Thus, H is coflasque. Additionally, since C is coflasque and Q is quasi-trivial, this extension splits $H \cong C \times Q$. Thus, the map

$$H^1(k, H) \to H^1(k, C)$$

is an isomorphism. Applying Theorem 1, we see that

$$Ж(k, E) \cong Ж(k, T).$$

Assume that T and T' are stably birational tori. Then, their flasque invariants coincide Colliot-Thélène and Sansuc 1977, Proposition 2.6 and there exist short exact sequences

$$1 \to P \to E \to T \to 1$$
$$1 \to P' \to E \to T' \to 1$$

with both P and P' quasi-trivial Colliot-Thélène and Sansuc 1977, Lemme 1.8. From the above, we see that

$$Ж(k, T) \cong Ж(k, E) \cong Ж(k, T').$$

Assume that T is retract rational. Then appealing to Theorem 4.2.1 we have an exact sequence

$$1 \to Q \to P \to T \to 1$$

where Q is invertible and P is quasi-trivial. Thus, $Ш(-, T) \cong Ш(-, P)$. Since P is quasi-trivial it is coflasque so $Ш(-, P) = H^1(-, P)$ by Theorem 1. Proposition 3.2.1 says the latter is trivial.

Assume $Ш(-, T)$ is trivial. From Proposition 3.5.1, C is special. Then, from Corollary 3.2.1 C is invertible. Then there is a quasi-trivial torus P with $P = C \times D$ so

$$1 \to D \to P \to C \to 1$$

is a flasque resolution with D invertible. Thus, Theorem 4.2.1 shows C is retract rational. □

From Proposition 3.5.1, understanding when $Ш(k, G)$ is trivial amounts to understanding when a coflasque algebraic group is special. When k is perfect and of cohomological dimension ≤ 1, then *all* torsors of connected algebraic groups are trivial by Serre's Conjecture I (now Steinberg's Theorem Serre 2002, §III.2.3). Thus, we have:

Proposition 3.5.3. If k is a field of cohomological dimension ≤ 1, then $Ш(k, G) = *$ for all reductive algebraic groups G. In particular, this holds for finite fields k.

In a more subtle manner, we may also leverage Serre's Conjecture II:

Conjecture 3.5.1 (Serre's Conjecture II)**.** If k is a perfect field of cohomological dimension ≤ 2, then $H^1(k, G) = *$ for all simply-connected semisimple algebraic groups.

Note that Serre's conjecture II is still open in general, although many cases are known (see the survey Gille 2010). In particular, the conjecture is proved for non-archimedean local fields Kneser 1965a; Kneser 1965b.

Proposition 3.5.4. Suppose k is a field for which the conclusion of Serre's Conjecture II holds. Let C be a coflasque reductive algebraic group over k and consider the exact sequence

$$1 \to C^{sc} \to C \to C^{tor} \to 1$$

where C^{sc} is the derived subgroup of C and C^{tor} is the abelianization. Then the induced map $H^1(k, C) \to H^1(k, C^{tor})$ is injective.

Proof. By definition, the derived subgroup C^{sc} of a coflasque reductive algebraic group is semisimple simply-connected. Since any form of a simply-connected semisimple algebraic group is simply-connected semisimple, all fibers of the map $H^1(k, C) \to H^1(k, C^{tor})$ are trivial or empty. $\square$

In the remainder of this section, our goal is to understand $Ш(k, G)$ over number fields. We begin with characterizations of coflasque algebraic groups over local fields.

Lemma 3.5.1. If C is a coflasque algebraic group over a nonarchimedean local field k, then $H^1(k, C) = *$.

Proof. By Proposition 3.5.4, it suffices to assume C is a coflasque torus. Let K/k be any Galois splitting field of C with Galois group $\Gamma_{K/k}$. From Tate-Nakayama duality, see e.g. Voskresenskiĭ 1998, Theorem 11.3.5, we have an isomorphism

$$H^1(\Gamma_{K/k}, C(K)) \cong H^1(\Gamma_{K/k}, \widehat{C}) = 0$$

since $\widehat{C}$ is coflasque. Thus $H^1(k, C) = 0$ as desired. $\square$

The archimedean case is more complicated. For real tori, the notions of flasque, coflasque, and quasi-trivial all coincide, so $H^1(\mathbb{R}, T) = *$ for a coflasque real torus T. However, coflasque real algebraic groups can have non-trivial torsors. Thus $Ш(\mathbb{R}, G)$ may be non-trivial when G is a not a torus.

Example 3.5.1. The group $\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}(5,1)$ is simply-connected hence coflasque. However,

$$|H^1(\mathbb{R}, \mathrm{SL}_2(\mathbb{H}))| = 2$$

from Adams and Taïbi 2018, Section 10.1. Similarly, for the compact form of E_8, which is also simply-connected, we have

$$|H^1(\mathbb{R}, E_8)| = 3$$

from Adams and Taïbi 2018, Section 10.2.

Nevertheless, from Borovoĭ 1988, the set $H^1(\mathbb{R}, G)$ has an explicit combinatorial description for any reductive algebraic group G, so this case can be explicitly computed.

3.5.2 Number fields

We recall the *Tate-Shafarevich group* of a linear algebraic group (see, e.g., Platonov and Rapinchuk 1994, §7). If G is a reductive algebraic group and k is a number field, then

$$\text{Ш}^i(k, G) := \ker\left(H^i(k, G) \to \prod_v H^i(k_v, G_{k_v})\right),$$

where the product is over all places v of k. The *Tate-Shafarevich group* is the case where $i = 1$, which is an abelian group even if G is not commutative.

For simply-connected algebraic groups, the Tate-Shafarevich group is trivial. In fact, we have the following even stronger result Platonov and Rapinchuk 1994, Theorem 6.6:

Theorem 3.5.1 (Kneser, Harder, Chernousov)**.** If G is a simply-connected semisimple algebraic group over a number field k, then the natural map

$$H^1(k, G) \to \prod_{v \text{ real}} H^1(k_v, G_{k_v})$$

is a bijection.

Lemma 3.5.2. Colliot-Thélène 2008, Proposition 9.4(ii) Let G be a reductive algebraic group over a number field. Suppose

$$1 \to S \to H \to G \to 1$$

is a flasque resolution of G. Then the connecting homomorphism induces a bijection $Ш^1(G) \cong Ш^2(S)$.

Finally, we are in a position to prove our final result:

Proof of Theorem 3. Let

$$1 \to P \to C \to G \to 1$$

be a coflasque resolution of G and let

$$1 \to S \to H \to G \to 1$$

be a flasque resolution of G. Setting $H' := C \times_G H$, we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & P \arrow[r, equal] \arrow[d] & P \arrow[d] & \\
1 \arrow[r] & S \arrow[r] \arrow[d, equal] & H' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd} \tag{3.5.1}$$

where P is a quasi-trivial torus, S is a flasque torus, H' and H are quasi-trivial algebraic groups and C is a coflasque algebraic group. By Lemma 3.5.2, the induced maps

$$Ш^1(k, G) \to Ш^2(k, S)$$

and

$$Ш^1(k, C) \to Ш^2(k, S)$$

are isomorphisms. By Proposition 3.2.4, there is a flasque resolution of the first type

$$1 \to S \to (H')^{tor} \to C^{tor} \to 1.$$

Using Lemma 3.5.2 again, the induced map $Ш^1(k, C^{tor}) \to Ш^2(k, S)$ is an isomorphism. Thus the morphism $Ш^1(k, C) \to Ш^1(k, C^{tor})$ is an isomorphism.

The task is to compute $\text{Ж}(k, G)$, Since $\text{Ж}(k, G) \cong H^1(k, C)$ by Theorem 3.4.1, we must compute $H^1(k, C)$. We start with the short exact sequence

$$1 \to C^{sc} \to C \to C^{tor} \to 1\ .$$

We get a commutative diagram

$$\begin{tikzcd}
H^1(k, C^{sc}) \arrow[r] \arrow[d] & H^1(k, C) \arrow[r] \arrow[d] & H^1(k, C^{tor}) \arrow[d] \\
\prod_v H^1(k_v, C^{sc}) \arrow[r] & \prod_v H^1(k, C) \arrow[r] & \prod_v H^1(k_v, C^{tor})
\end{tikzcd}$$

From Lemma 3.5.1, we have

$$H^1(k_v, C) = H^1(k_v, C^{sc}) = H^1(k_v, C^{tor}) = *$$

for any finite v; the same holds for complex v. Since coflasque tori are quasi-trivial over $\mathbb{R}$, we know $H^1(\mathbb{R}, C^{tor}) = *$. Thus, we reduce to the commutative diagram

$$\begin{tikzcd}
H^1(k, C^{sc}) \arrow[r] \arrow[d, equal] & H^1(k, C) \arrow[r] \arrow[d] & H^1(k, C^{tor}) \arrow[d] \\
\prod_{v \text{ real}} H^1(k_v, C^{sc}) \arrow[r] & \prod_{v \text{ real}} H^1(k_v, C) \arrow[r] & *
\end{tikzcd}$$

with exact rows, where the left vertical map is a bijection by Theorem 3.5.1. In particular, the map

$$H^1(k, C) \to \prod_{v \text{ real}} H^1(k_v, C)$$

is surjective, and we obtain the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \text{Ш}^1(k,C) \arrow[r] \arrow[d, equal] & H^1(k,C) \arrow[r] \arrow[d] & \prod\limits_{v \text{ real}} H^1(k_v,C) \arrow[r] \arrow[d] & * \\
1 \arrow[r] & \text{Ш}^1(k,C^{tor}) \arrow[r, equal] & H^1(k,C^{tor}) \arrow[r] & 1 &
\end{tikzcd}$$

with exact rows. We have a surjective function $H^1(k,C) \to \text{Ш}^1(k,C^{tor})$ that has a canonical retract.

For any cocycle $\gamma \in Z^1(k,C)$, the twisted group ${}^{\gamma}C^{tor}$ is isomorphic to C^{tor}. Thus $\text{Ш}^1(k,C) \cong \text{Ш}^1(k,{}^{\gamma}C)$ and we conclude all fibers of the map

$$H^1(k,C) \to \prod_{v \text{ real}} H^1(k_v,C)$$

are isomorphic. Using Theorem 1 and the isomorphism $\text{Ш}^1(k,G) \cong \text{Ш}^1(k,C)$, established above, we can rewrite the resulting direct product

$$\text{Ж}(k,G) \cong H^1(k,C) \cong \text{Ш}^1(k,C) \times \prod_{v \text{ real}} H^1(k_v,C) \cong \text{Ш}^1(k,G) \times \prod_{v \text{ real}} \text{Ж}(k_v,G).$$

$\square$

Chapter 4

Consequences of Exceptional Collections in Arithmetic and Rationality

This chapter was joint work with Dr. Matthew Robert Ballard, Dr. Alexander Duncan, and Dr. Patrick McFaddin.

4.1 Introduction

Developments over the past forty years have established derived categories of coherent sheaves as a versatile language for capturing deep but obscure geometric connections between different algebraic varieties. Central to these investigations has been the tie between rationality questions and derived categories.

A basic motivating question is the following: to what extent can the derived category be used as tool to understand the rationality of a variety?

Examples in low dimension provide some insight. For a smooth projective curve C over a field k, the bounded derived category $\mathrm{D^b}(C) = \mathrm{D^b}(\operatorname{coh} C)$ of coherent sheaves on C admits a full k-exceptional (or étale-exceptional; see Definition 1.4.2) collection if and only if $C \cong \mathbb{P}^1_k$.

Over a perfect field k, the derived category of a smooth rational projective surface always has a full étale-exceptional collection, though not a full k-exceptional collection in general. This follows from the classification of minimal rational surfaces; see for example Manin and Tsfasman 1986, and a case by case analysis for del Pezzo varieties Auel and Bernardara 2018.

More generally, it is expected that rationality of X should guarantee that $\mathrm{D}^{\mathrm{b}}(X)$ admits a semi-orthogonal decomposition into components that are not too complicated. A precise conjecture in Bernardara and Bolognesi 2012b states that each component should embed admissibly into the derived category of a smooth projective variety of dimension at most $\dim X - 2$. The structure of derived categories of Fano threefolds over $\mathbb{C}$ provides evidence for this belief Kuznetsov 2016. Kuznetsov's conjecture on the rationality of a cubic fourfold also follows along this general belief Kuznetsov 2010.

In the converse direction, Vial showed that any geometrically rational smooth projective surface with a full (numerical) k-exceptional collection is k-rational Vial 2017. Brown and Shipman showed that a smooth complex projective surface with a full strong exceptional collection of line bundles is rational Brown and Shipman 2017.

More generally, a conjecture of Orlov states that a smooth projective variety with a full exceptional collection is rational. Even stronger, Lunts conjectures that over a general field k a full k-exceptional collection for X implies that X admits a locally-closed stratification into subvarieties that are each k-rational Elagin and Lunts 2016.

If we move beyond k-exceptionality to étale-exceptionality, then Auel and Bernardara conjecture that a smooth projective surface over k with a full étale-exceptional collection is actually k-rational Auel and Bernardara 2017. In dimension three, Bernardara and Bolognesi ask whether for a smooth projective threefold with negative Kodaira dimension the existence of a semi-orthogonal decomposition into components admissibly embeddable into derived categories of curves is equivalent to rationality Bernardara and Bolognesi 2012b. Over $\mathbb{C}$ this is the case for conic bundles over minimal rational surfaces Bernardara and Bolognesi 2013.

This article provides more evidence to help answer these questions, both affirmatively and negatively. The first main result is that in dimension greater than three the existence of a full étale-exceptional collection says very little about the rationality properties of a variety.

Theorem 4. For any $d \geq 3$, there exists infinitely-many smooth projective geometrically rational d-folds Y over $\mathbb{Q}$ each of which has no $\mathbb{Q}$-points but admits a semi-orthogonal decomposition into derived categories of smooth points.

Following the previous chapter of this document, we introduce a new invariant $\text{Ж}(k, G)$ of a linear algebraic k-group G which exactly captures the G-torsors that cannot be detected by any Brauer groups of extension fields of k.

Given an object E of $\mathrm{D}^{\mathrm{b}}(X)$, where X is a smooth projective compactification of G satisfying (1) $\mathrm{End}_X(E)$ is a separable field extension of k, (2) $\mathrm{Ext}^i_X(E, E) = 0$ for $i \neq 0$, and (3) $E^{\oplus r}$ is G-linearizable, then twisting $E^{\oplus r}$ provides a cohomological invariant of G which lands in a Brauer group. Thus, when we twist such linearizable collection by torsors invisible to Brauer groups, the collection remains étale-exceptional.

Consequently, to produce examples of varieties with full étale-exceptional collections but without k-points, we just need to locate a smooth projective compactification X of a linear algebraic group G with $\text{Ж}(k, G) \neq *$ and such that $\mathrm{D}^{\mathrm{b}}(X)$ possesses a full étale-exceptional collection whose objects are linearizable up to passing to direct sums. We do this. The group G can either be the maximal torus for a form of the adjoint group of type A_3 or a norm-one torus for a biquadratic extension. The variety X comes from twisting the split toric variety associated to the Weyl chambers of A_3.

Theorem 4 warns us not too expect much from the assumption that the derived category of our variety decomposes into derived categories of smooth points. What if we assume the full collection is k-exceptional?

From the evidence recapitulated earlier, we see that little is known in general and even less known in higher dimensions. In particular, the answer is unknown for smooth projective varieties. Over $\mathbb{C}$, any such variety is immediately rational. However, over a general field, toric varieties need not even possess a k-point; a fact useful in Theorem 4. Moreover, even when they possess a k-point, toric varieties need not even be retract rational much less rational. Thus, the class of toric varieties is an obvious testing ground for the conjecture.

For smooth projective toric varieties with a point over the ground field k, we verify that possession of a full k-exceptional collection implies k-rationality.

Theorem 5. Let X be a smooth projective toric variety over a field k with $X(k) \neq \emptyset$. If $\mathrm{D}^{\mathrm{b}}(X)$ has a full k-exceptional collection, then X is k-rational.

Theorem 5 follows from known results about how the structure of the Picard group, as Galois module, controls rationality questions for toric varieties.

Acknowledgments

The authors would like to thank B. Antieau for several helpful comments. Via the first author, this material is based upon work supported by the National Science Foundation under Grant No. NSF DMS-1501813. Via the second author, this work was supported by a grant from the Simons Foundation (638961, AD). The third author was partially supported by a USC SPARC grant. The fourth author was partially supported by an AMS-Simons travel grant

Notation

Throughout, k denotes an arbitrary field with separable closure $\overline{k}$. A variety is an integral separated scheme of finite type over a field. A linear algebraic group is a smooth affine group scheme of finite-type over k. For a k-variety X and a field extension L/k, we write $X_L := X \times_{\operatorname{Spec} L} \operatorname{Spec} k$ and $\overline{X} := X_{\overline{k}}$. Let Γ_k be the absolute

Galois group $\mathrm{Gal}(\overline{k}/k)$, which is a profinite group. Given a Γ_k-lattice Q, we denote the Cartier dual torus by $\mathscr{D}(Q)$.

4.2 Background

4.2.1 Preliminaries on lattices

We recall some (mostly standard) facts about Γ-lattices; see, for example, Colliot-Thélène and Sansuc 1977 or Voskresenskiĭ 1998.

Definition 4.2.1. Let Γ be a profinite group and let M be a Γ-lattice. Note that the image of the Γ-action factors through a finite group G called the *decomposition group*, which acts faithfully on M.

1. M is *permutation* if there is a $\mathbb{Z}$-basis of M permuted by Γ.
2. M is *stably permutation* if there exist permutation lattices P_1 and P_2 such that $M \oplus P_1 = P_2$.
3. M is *invertible* if it is a direct summand of a permutation lattice.
4. M is *quasi-permutation* if there exists a short exact sequence
$$0 \to M \to P_1 \to P_2 \to 0$$
where P_1 and P_2 are permutation lattices.

Given a Γ-lattice M, the *dual lattice* $M^\vee := \mathrm{Hom}_{\mathsf{Ab}}(M, \mathbb{Z})$ is the set of group homomorphisms from M to $\mathbb{Z}$ with the natural Γ-action where $\mathbb{Z}$ has the trivial Γ-action. Note that this duality induces an exact anti-equivalence of the category of Γ-lattices with itself.

4.2.2 Birational geometry of tori over general fields

Let Γ_k be the absolute Galois group of the field k.

Definition 4.2.2. A k-*torus* is an algebraic group T over k such that $T_{\bar{k}} \cong \mathbb{G}_m^n$. A torus is *split* if $T \cong \mathbb{G}_m^n$. A field extension L/k satisfying $T_L \cong \mathbb{G}_m^n$ is called a *splitting field* of the torus T. Any torus admits a finite Galois splitting field.

Recall that there is an anti-equivalence of categories between Γ_k-lattices and k-tori, which we will call *Cartier duality* (see, e.g., Voskresenskiĭ 1998). Given a torus T, the Cartier dual (or *character lattice*) $\widehat{T}$ is the Γ-lattice $\mathrm{Hom}_{\bar{k}}(\overline{T}, \mathbb{G}_{m,\bar{k}})$. Given a Γ_k-lattice M, we use $\mathscr{D}(M)$ to denote the Cartier dual torus.

An étale algebra over k of degree n is a commutative separable algebra over k of dimension n. In other words, $E = F_1 \times \cdots \times F_r$ where $F_1, \ldots, F_r$ are separable field extensions of k. There is an antiequivalence between finite Γ_k-sets Ω and étale algebras E via

$$\Omega = \mathrm{Hom}_{k-\mathsf{Alg}}(E, \bar{k}) \text{ and } E = \mathrm{Hom}_{\Gamma_k-\mathsf{Set}}(\Omega, \bar{k})$$

with the natural Γ_k-action and k-algebra structure on $\bar{k}$ (see, e.g., Knus et al. 1998, §18).

Recall that any étale algebra E over k has a *norm map* $N : E^\times \to k^\times$. We obtain an exact sequence

$$1 \to R^{(1)}_{E/k}\mathbb{G}_m \to R_{E/k}\mathbb{G}_m \to \mathbb{G}_m \to 1$$

of tori over k where the torus $R^{(1)}_{E/k}\mathbb{G}_m$ is called the *norm-one torus* of the extension E/k.

Definition 4.2.3. A k-variety X is *rational* if X is birationally equivalent to $\mathbb{A}^n_k$ for some $n \geq 0$. We say X is *stably rational* if $X \times \mathbb{A}^n_k$ is birational to $\mathbb{A}^m_k$ for some $n, m \geq 0$. We say X is *retract rational* if there is a dominant rational map $f : \mathbb{A}^n_k \dashrightarrow X$ that has a rational section $s : X \dashrightarrow \mathbb{A}^n_k$ such that $f \circ s$ is the identity on X.

Rationality properties of T are encoded by its flasque resolutions.

Theorem 4.2.1. If $T = R^{(1)}_{K/k}\mathbb{G}_m$ is a norm-one torus corresponding to a Galois extension K/k with Galois group G, then T is retract rational if and only if all Sylow subgroups of G are cyclic.

Proof. This is a combination of Colliot-Thélène and Sansuc 1977, Proposition 1.2 and Saltman 1984, Theorem 3.14, see also Endô and Miyata 1975. □

It is not known whether there exists a stably rational torus that is not rational. But, a result of Voskresenskiĭ allows one to deduce rationality when the flasque resolution is split.

Theorem 4.2.2. If T fits into a exact sequence of tori

$$1 \to \mathbb{G}_m^r \to R_{E/k}\mathbb{G}_m \to T \to 1$$

where E is étale k-algebra, then T is rational.

Proof. This is Voskresenskiĭ 1971, Theorem 2. □

4.2.3 Cohomological invariants and torsors cloaked from Brauer groups

We review the notion of a *cohomological invariant* following Serre 2003.

Let Fields_k be the category of field extensions over k, Grp the category of groups, Ab the category of abelian groups, Sets the category of sets, and Sets_* the category of pointed sets.

Fixing a base field k, we consider two functors

$$A : \mathsf{Fields}_k \to \mathsf{Sets}_*$$

and

$$H : \mathsf{Fields}_k \to \mathsf{Ab}\ .$$

A *normalized H-invariant of A* is a morphism of functors $A \to H$. The group of all such invariants will be denoted $\mathrm{Inv}_k(A, H)_{norm}$.

Remark 4.2.1. We demand a priori that A is a functor into *pointed* sets. This explains the adjective "normalized." This condition is harmless as a general H-invariant of A can be written uniquely as the sum of a normalized invariant and a "constant" invariant coming from $H(k)$.

The two kinds of functors we will consider are as follows. Given an algebraic group G over k, we may view Galois cohomology

$$H^i(-, G) : \mathsf{Fields}_k \to \mathsf{Sets}_*$$

as a functor (the codomain may be interpreted as Grp if $i = 0$ or Ab if G is commutative).

Given an étale algebra E over k we may consider the Brauer group

$$\mathrm{Br}_E := \mathrm{Br}(- \otimes_k E) : \mathsf{Fields}_k \to \mathsf{Ab}$$

as a functor. Note that we have a canonical isomorphism

$$\mathrm{Br}(- \otimes_k E) \cong H^2(-, R_{E/k}\mathbb{G}_m).$$

Definition 4.2.4. We set

$$\text{Ж}(k, G) = \bigcap_E \bigcap_\alpha \ker \Big(\alpha(k) : H^1(k, G) \to \mathrm{Br}(E)\Big)$$

where the intersections run over all étale algebras E and all normalized cohomological invariants α.

One of the main results of the previous chapter characterizes $\text{Ж}(k, G)$ in terms of a coflasque resolution of G. We recall some facts about $\text{Ж}(k, G)$.

Proposition 4.2.1. Let S be a torus over k. The invariant $\text{Ж}(-, S)$ is a stable birational invariant of S. Moreover, S is retract rational if and only if $\text{Ж}(-, S)$ is trivial.

Proof. This is Ballard et al. 2019, Proposition 5.2. □

Next, we will need a computation of Ж for a particular norm-one torus.

Proposition 4.2.2. Assume k is a number field. Let L/k be a biquadratic extension which is cyclic at every place. Then

$$\text{Ж}(k, R^{(1)}_{L/k}\mathbb{G}_m) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. From Ballard et al. 2019, Theorem 3, we have

$$\text{Ж}(k, T) \cong \text{Ш}^1(k, T)$$

for a torus over a number field. From Voskresenskiĭ 1998, Theorems 11.5, 11.6, we reduce to checking that

$$H^1(\Gamma, \widehat{R^{(1)}_{L/k}\mathbb{G}_m}) = \mathbb{Z}/2\mathbb{Z}.$$

This computation is standard, see e.g. Colliot-Thélène and Sansuc 1977, Proposition 1.1. □

Finally, we recall a fact about biquadratic extensions over $\mathbb{Q}$.

Proposition 4.2.3. There exists infinitely many non-isomorphic biquadratic extensions $L/\mathbb{Q}$ cyclic at every place.

Proof. For a biquadratic extension, we have $L = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ for coprime a and b. Let us take a and b prime for simplicity. Any extension is cyclic over $\mathbb{R}$ or $\mathbb{C}$ so we reduce to primes.

To guarantee that it is cyclic at every prime p, we need one of a, b, ab to be a square in $\mathbb{Q}_p$. We can pick a so that it is a square in $\mathbb{Q}_2$. For odd $p \neq a, b$, we reduce to checking that a, b, or ab is a square in $\mathbb{Z}/p\mathbb{Z}$ by Hensel's Lemma. Multiplicativity of the Legendre symbol guarantees that one of a, b, ab is then always a square. Finally, we need to choose b to be square modulo a. We see there are infinitely many choices for the pair (a, b). □

4.2.4 Derived categories, semi-orthogonal decompositions, and exceptional collections

Recall from Chapter 1 the conventions used for semiorthogonal decompositions and exceptional collections. For a triangulated category T, we use the standard notation $\mathrm{Ext}^n_{\mathsf{T}}(A,B) = \mathrm{Hom}_{\mathsf{T}}(A,B[n])$. For objects A,B of $\mathrm{D^b}(X)$, we use $\mathrm{End}_X(A)$ and $\mathrm{Ext}^n_X(A,B)$ to denote $\mathrm{End}_{\mathrm{D^b}(X)}(A)$ and $\mathrm{Ext}^n_{\mathrm{D^b}(X)}(A,B)$, respectively.

Lemma 4.2.1. A triangulated category is categorically representable in dimension 0 if and only if it possesses a full étale exceptional collection.

Proof. The only irreducible smooth projective varieties of dimension zero are points: $\operatorname{Spec} L$ for a finite separable field extension L/k. □

In Bernardara and Bolognesi 2012b, Question 4.5.2, the following question is posed:

Question 4.2.1. For a smooth projective threefold X with Kodaira dimension $\kappa < 0$, is X rational if and only if X is categorically representable in codimension 2?

4.3 Twisting by a torsor and its effect on the derived category

Let G be a linear algebraic group over k and X a smooth projective variety over k with G-action. Let $p, s : G \times X \to X$ be the projection and action morphisms, respectively. Let $\mu : G \times G \to G$ denote the group multiplication.

Definition 4.3.1. Let E be an object of $\mathrm{QCoh}(X)$. A *G-linearization* of E is an isomorphism $\theta : s^*E \xrightarrow{\sim} p^*E$ which satisfies the cocycle condition given by commutativity of the diagram below for the various maps $G \times G \times X \to G \times X$.

$$\begin{tikzcd}
(s\circ(1_G\times s))^*E \arrow[r, "(1_G\times s)^*\theta"] \arrow[d, equal] & (p\circ(1_G\times s))^*E \arrow[r, "1\times p^*\theta"] & (p\circ(1_G\times p))^*E \arrow[d, equal] \\
(s\circ(\mu\times 1_X))^*E \arrow[rr, "(\mu\times 1_X)^*\theta"] & & (p\circ(\mu\times 1_X))^*E
\end{tikzcd}$$

A G-*equivariant sheaf* is a pair (E,θ) consisting of a quasi-coherent sheaf together with a G-linearization. We also call such a pair (E,θ) a G-*lift* of E. A morphism $f:(E,\theta)\to(F,\upsilon)$ is a morphism of quasi-coherent sheaves $f:E\to F$ so that

$$\begin{tikzcd}
s^*E \arrow[r, "s^*f"] \arrow[d, "\theta"'] & s^*E \arrow[d, "\upsilon"] \\
p^*F \arrow[r, "p^*f"] & p^*E
\end{tikzcd}$$

commutes. We refer the reader to Ballard, Favero, and Katzarkov 2014 for more information on categories of equivariant sheaves, e.g., pushforwards and pullbacks via equivariant morphisms.

Definition 4.3.2. Let U be a (right) G-torsor and X a G-variety. The *twist by* U or *contracted product* ${}^UX$ is the quotient of $U\times X$ via the left G-action

$$g\cdot(u,x):=(ug^{-1},gx).$$

We let $t:U\times X\to {}^UX$ denote the quotient map.

Let U be a (right) G-torsor. Let U^{ad} be the automorphism group of U as a G-variety (often called the *gauge group* of U). We may give U the structure of a (U^{ad},G)-bitorsor; on geometric points $f\in U^{\mathrm{ad}}(\overline{k}), g\in G(\overline{k}), u\in U(\overline{k})$ we have $(f,g)\cdot u=f(u)g$. There is also an opposite bitorsor U^{op}, which is a (G,U^{ad})-bitorsor with the same underlying space U but $(g,f)\cdot u=f^{-1}(u)g^{-1}$ on geometric points $f\in U^{\mathrm{ad}}(\overline{k}), g\in G(\overline{k}), u\in U(\overline{k})$. Note that the contracted product $U^{op}\times^{U^{\mathrm{ad}}}U$ is isomorphic to the canonical (G,G)-bitorsor structure on the group G.

For the trivial G-torsor, we have already identified the map inducing the quotient. Indeed, if we equip $G \times X$ with an action $g \cdot (u, x) \mapsto (ug^{-1}, gx)$, the map $s : G \times X \to X$ is the quotient. Under this action, the projection morphism p is G-equivariant. Similarly, for a G-torsor U the projection $q : U \times X \to X$ is G-equivariant.

Theorem 4.3.1. Let X be a G-variety, U a G-torsor and $Y = {}^{U}X$. There is an equivalence of categories

$$\Psi_U : \mathrm{QCoh}_G(X) \to \mathrm{QCoh}_{U^{\mathrm{ad}}}(Y) \ ,$$

which is naturally isomorphic to the identity when U is a trivial G-torsor.

Proof. Consider the diagram

$$\begin{tikzcd} & U \times X \arrow[dl, "q"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}$$

There is a left $(U^{\mathrm{ad}} \times G)$-action on $U \times X$ via

$$(f, g) \cdot (u, x) = (f(u)g^{-1}, gx)$$

on geometric points. Giving X the trivial U^{ad}-action and Y the trivial G-action, the diagram above is $(U^{\mathrm{ad}} \times G)$-equivariant. The map Ψ_U is thus constructed as the composition

$$\mathrm{QCoh}_G(X) \xrightarrow{q^*} \mathrm{QCoh}_{U^{\mathrm{ad}} \times G}(U \times X) \xrightarrow{(t_* -)^G} \mathrm{QCoh}_{U^{\mathrm{ad}}}(Y).$$

Assume that U is split and pick an isomorphism $U \cong G$. We can then assume that $t = s$ and $q = p$. For $E \in \mathrm{QCoh}_G(X)$, there is a linearization $\theta : s^*E \to p^*E$, which is equivariant for the left action of $U^{\mathrm{ad}} = G$ on $G \times X$. Via adjunction we have a U^{ad}-equivariant morphism $E \to s_*p^*E$ which is the composition

$$E \xrightarrow{\eta_E} s_*s^*E \xrightarrow{s_*\theta} s_*p^*E,$$

where η_E is the unit of adjunction. Taking G-invariants gives

$$E \overset{\nu_E}{\to} (s_* s^* E)^G \xrightarrow{(s_*\theta)^G} (s_* p^* E)^G,$$

where the latter map is the image of an isomorphism under a functor and is hence an isomorphism. We have

$$s_* s^* E \cong E \otimes k[G].$$

Under this isomorphism, $E \to s_* s^* E$ equals $E \to E \otimes k \subseteq E \otimes k[G]$ which is exactly the G-invariants. Therefore, the map

$$E \to (s_* p^* E)^G$$

is an isomorphism.

Returning to general U, we have a $(U^{\mathrm{ad}} \times G)$-action on $U^{\mathrm{op}} \times U \times X$. Taking the quotient first by G and then by U^{ad} gives the composition $\Psi_{U^{\mathrm{op}}} \circ \Psi_U$. We can however take the quotient by U^{ad} to yield an action of G on $G \times X$. Thus,

$$\Psi_{U^{\mathrm{op}}} \circ \Psi_U \cong \Psi_G \cong \mathrm{Id}\,.$$

□

We now check that for any G-torsor U, the sheaf $\Psi_U(E)$ is a form of E. Fix an extension L/k with a point $u \in U(L)$. This gives an isomorphism

$$\begin{aligned} \psi : G_L &\to U_L \\ g &\mapsto g \cdot u. \end{aligned}$$

By smooth descent, there exists an isomorphism $f : X_L \to Y_L$ making the square

$$\begin{tikzcd} G_L \times X_L \arrow[r, "\psi \times 1"] \arrow[d, "s"'] & U_L \times X_L \arrow[d, "t"] \\ X_L \arrow[r, "f"] & Y_L \end{tikzcd}$$

commute. Having fixed an isomorphism between X_L and Y_L, we need to check that $f^*\Psi_U(E)_L \cong E_L$.

Proposition 4.3.1. For a G-equivariant sheaf (E, θ), a G-torsor U and the above isomorphism $f : X_L \to Y_L$, there is a natural isomorphism $\vartheta : f^*\Psi_U(E)_L \xrightarrow{\sim} E_L$.

Proof. Recall that $\Psi_U(E) = (t_*q^*E)^G$ and hence

$$f^*\Psi_U(E)_L = (s_*(\psi \times 1)^*t^*(t_*q^*E_L)^G)^G \cong (s_*(\psi \times 1)^*q^*E_L)^G$$

since $t^*(t_*-)^G \cong 1$. We have a commutative diagram

$$\begin{tikzcd}
G_L \times X_L \arrow[rr, "\psi \times 1"] \arrow[dr, "p"'] & & U_L \times X_L \arrow[dl, "q"] \\
& X_L &
\end{tikzcd}$$

so that $s_*(\psi \times 1)^*q^*E_L = s_*p^*E_L$. Consider the map $E \to s_*p^*E$ arising from the isomorphism $\theta : s^*E \to p^*E$ via adjunction. This map is the composition

$$E \xrightarrow{\nu_E} s_*s^*E \xrightarrow{s_*\theta} s_*p^*E,$$

where ν_E is the unit of adjunction. Since θ is an isomorphism, so is $s_*\theta$. We have another commutative diagram

$$\begin{tikzcd}
G \times X \arrow[rr, "\iota"] \arrow[dr, "s"'] & & G \times X \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

where $\iota(g, x) = (g^{-1}, g \cdot x)$ is an involution. Then

$$(s_*s^*E)^G = (p_*\iota_*\iota^*p^*E)^G \cong (p_*p^*E)^G \cong (E \otimes k[G])^G = E.$$

Thus ν_E is also an isomorphism, proving the above claim. $\square$

Let g_τ be the unique element of $G(L)$ satisfying ${}^\tau u = g_\tau u$. This gives a 1-cocycle in $H^1(k, G)$. Inspecting the proof of Proposition 4.3.1, we can explicitly identify the 1-cocycle corresponding to the form $\Psi_U(E)$ of E.

Corollary 4.3.1. The cocycle in $H^1(k, \mathrm{Aut}(E))$ corresponding to $\Psi_U(E)$ is $\tau \mapsto \theta_{g_\tau}$.

Proof. We have a twisted Γ-action on X_L given by $\tau \mapsto f^{-1} \circ {}^\tau f$ and we need to identify the Γ-equivariant structure on E_L coming from the isomorphism in Proposition 4.3.1.

Up top working G-equivariantly, if we conjugate τ we obtain

$$(\psi \times 1)^{-1} \circ \tau \circ (\psi \times 1)(g, x) = (g_\tau {}^\tau g, {}^\tau x).$$

Thus, the new Γ-action on $G_L \times X_L$ is $(g, x) \mapsto g_\tau \cdot {}^\tau(g, x)$. Using the inclusion of a slice

$$X_L \to G_L \times X_L$$
$$x \mapsto (1, x)$$

we see that the twisted Γ-action on X_L is given by $x \mapsto g_\tau \cdot {}^\tau x$. Thus,

$$f^* \tau^* (f^{-1})^* E_L = \tau^* g_\tau^* E_L.$$

The twisted equivariant structure on E_L comes from the maps

$$\alpha_\tau : f^* \tau^* (f^{-1})^* E_L \overset{f^*\tau^*(f^{-1})^*\vartheta^{-1}}{\longrightarrow} f^* \tau^* \Psi_U(E)_L = f^* \Psi_U(E)_L \overset{\vartheta}{\to} E_L \tag{4.3.1}$$

where $\tau^* \Psi_U(E)_L = \Psi_U(E)_L$ since it is pulled back from k. From the proof of Proposition 4.3.1, we can identify

$$t^* \vartheta^{-1} : t^* (f^{-1})^* E_L = (\psi^{-1} \times 1)^* s^* E_L \overset{(\psi^{-1}\times 1)^*\theta}{\longrightarrow} (\psi^{-1} \times 1)^* p^* E_L = q^* E_L = t^* \Psi(E)_L.$$

We have already seen

$$\tau^* (\psi^{-1} \times 1)^* = (\psi^{-1} \times 1)^* \tau^* \mu_{g_\tau}^*$$

where

$$\mu_{g_\tau} : G \times X \to G \times X$$
$$(g, x) \mapsto (g_\tau g, x).$$

Then

$$\begin{aligned} t^*(f^{-1})^*\alpha_\tau &= (\psi^{-1} \times 1)^*\theta^{-1} \circ (\psi^{-1} \times 1)^*\tau^*\mu_{g_\tau}^*\theta \\ &= (\psi^{-1} \times 1)^*(\theta^{-1} \circ \tau^*\mu_{g_\tau}^*\theta) \\ &= (\psi^{-1} \times 1)^*\tau^*(\theta^{-1} \circ \mu_{g_\tau}^*\theta) \end{aligned}$$

since $\tau^*\theta = \theta$. To get back α_τ, we apply $f^*(t_*-)^G$, but

$$f^*(t_*-)^G = (s_*(\psi \times 1)^*-)^G$$

from flat base change. Hence,

$$\alpha_\tau = (s_*\tau^*(\theta^{-1} \circ \mu_{g_\tau}^*\theta))^G = \tau^*(s_*(\theta^{-1} \circ \mu_{g_\tau}^*\theta))^G$$

since s is pulled back from k. From the cocycle condition on θ, we have

$$\theta^{-1} \circ \mu_{g_\tau}^*\theta = j_{g_\tau}^*(1 \times s)^*\theta$$

where $j_g : \{g\} \times G \times X \to G \times G \times X$ is the inclusion. Moreover,

$$(s_*j_{g_\tau}^*(1 \times s)^*\theta)^G = (s_*s^*(i_{g_\tau}^*\theta))^G = i_{g_\tau}^*\theta =: \theta_{g_\tau}$$

where $i_g : \{g\} \times X \to G \times X$ is the inclusion. $\square$

Given two G-equivariant quasi-coherent sheaves (E, θ) and (F, ϕ), we get an action of G on the space of non-equivariant morphisms from E to F. Assume that E is coherent. We let a be the map making the diagram below commute

$$\begin{tikzcd} \mathrm{Hom}_X(E, F) \arrow[r, "s^*"] \arrow[d, "a"'] & \mathrm{Hom}_{G\times X}(s^*E, s^*F) \arrow[d, "\phi \circ (-) \circ \theta^{-1}"] \\ \mathrm{Hom}_X(E, F) \otimes_k k[G] & \mathrm{Hom}_{G\times X}(p^*E, p^*F) \arrow[l, "\sim"'] \end{tikzcd}$$

where the isomorphism comes from adjunction and the projection formula. Note that the natural map

$$\mathrm{Hom}_X(E, F) \otimes k[G] \to \mathrm{Hom}_X(E, F \otimes k[G])$$

is an isomorphism since E is coherent. Note that the action of a commutes with composition. In particular, we get an algebraic group homomorphism

$$a : G \to \mathrm{Aut}(\mathrm{End}_X(E)).$$

Proposition 4.3.2. The image of $[U]$ under

$$H^1(k, a) : H^1(k, G) \to H^1(k, \mathrm{Aut}(\mathrm{End}_X(E)))$$

represents the algebra $\mathrm{End}_{^UX}(\Psi_U(E))$.

Proof. Working as in Proposition 4.3.1, pulling back by $f : X_L \to Y_L$ gives an isomorphism of algebras

$$\phi : \mathrm{End}_X(E)_L \xrightarrow{(f^{-1})^*} \mathrm{End}^\bullet((f^{-1})^*E)_L \xrightarrow{\alpha(-)\alpha^{-1}} \mathrm{End}_{^UX}(\Psi(E))_L$$

where $\alpha := (f^{-1})^*\vartheta$. Tracing things out as in Corollary 4.3.1, we have

$$\phi^{-1\tau}\phi = \theta_{g_\tau} \circ g_\tau^*(-) \circ \theta_{g_\tau}^{-1}.$$

Thus, a cocycle representing $\mathrm{End}_{^UX}(\Psi(E))$ is given by

$$\tau \mapsto \theta_{g_\tau} \circ g_\tau^*(-) \circ \theta_{g_\tau}^{-1}$$

But from the definitions, we also have

$$H^1(k, a)[g_\tau] = \theta_{g_\tau} \circ g_\tau^*(-) \circ \theta_{g_\tau}^{-1}.$$

□

Next, we show how to build a normalized cohomological invariant from an exceptional object that is linearizable up to sums.

Proposition 4.3.3. Let E be an étale-exceptional object in $\mathsf{D}^{\mathsf{b}}(X)$ such that $E^{\oplus r}$ is G-linearizable. Let $L = \mathrm{End}_X(E)$. The map

$$\begin{aligned} \varphi_E : H^1(-, G) &\to H^2(-, R_{L/k}\mathbb{G}_m) \\ U &\mapsto \left[\mathrm{End}_{{}^U X}(\Psi_U(E^{\oplus r}))\right] \end{aligned}$$

is a degree 2 normalized cohomological invariant.

Proof. The map is clearly a natural transformation of functors. Since we assumed that E is étale-exceptional, it is normalized. □

4.4 Exceptional collections and rationality for toric varieties

4.4.1 k-exceptional collections and rationality for toric varieties

We begin by introducing toric varieties defined over arbitrary fields. These varieties have been treated in Duncan 2016a, Elizondo et al. 2014b, Merkurjev and Panin 1997, Voskresenskii and Klyachko 1984, and are often called *arithmetic toric varieties*.

Definition 4.4.1. Given a torus T, a *toric T-variety* is a normal variety with a faithful T-action and a dense open T-orbit. A toric T-variety is *split* if T is a split torus. A *splitting field* of a toric T-variety is a splitting field of T.

Definition 4.4.2. A a toric T-variety whose dense open T-orbit contains a k-rational point is called *neutral* Duncan 2016a (or a *toric T-model* Merkurjev and Panin 1997). An orbit of a split torus always has a k-point, so a split toric variety is neutral; but the converse is not true in general.

To check if a toric variety has a rational point, it suffices to check on the open dense orbit.

Proposition 4.4.1. Let X be a smooth projective toric T-variety over a field k with dense open T-orbit U. Then $X(k) \neq \emptyset$ if and only if $U(k) \neq \emptyset$.

Proof. See Voskresenskii and Klyachko 1984, Proposition 4. $\square$

Remark 4.4.1. In what follows, we will use the term *toric variety* to mean toric T-variety for some fixed torus T with specified action, even though such a variety may have a toric structure for various tori. Note that care must be taken when referring to a *k-form* of a toric variety, since the torus T may not act on the k-form in general. We will typically twist by a T-torsor, so this subtlety will typically not arise. Note that any k-form of a toric variety is a toric variety (albeit for a potentially different torus action). We refer interested readers to Duncan 2016a for such considerations.

Let us review how to obtain arbitrary forms of toric varieties from the split case (see, for example, Voskresenskiĭ 1982; Elizondo et al. 2014b). Let T be the split torus of a split smooth projective toric variety X with fan Σ in the space $N \otimes \mathbb{R}$ associated to the lattice N. Let $\mathrm{Aut}(\Sigma)$ denote the subgroup of elements $g \in \mathrm{GL}(N)$ such that $g(\sigma) \in \Sigma$ for every cone $\sigma \in \Sigma$. There is a natural inclusion of $T \rtimes \mathrm{Aut}(\Sigma)$ into $\mathrm{Aut}(X)$ as the subgroup leaving the open orbit T-invariant.

The natural map

$$H^1(k, T \rtimes \mathrm{Aut}(\Sigma)) \to H^1(k, \mathrm{Aut}(X))$$

in Galois cohomology is surjective; the failure of this map to be a bijection amounts to the fact that there may be several non-isomorphic toric variety structures on the same variety (see Duncan 2016a for more details).

Suppose $X' = {}^{\gamma}X$ is a twisted form of a split toric variety for a cocycle γ representing a class in $H^1(k, T \rtimes \mathrm{Aut}(\Sigma))$. There is a "factorization" $X' = {}^{\alpha}({}^{\beta}X)$ where

β represents a class in $H^1(k, \mathrm{Aut}(\Sigma))$ and α represents a class in $H^1(k, ({}^{\beta}T))$. Informally, β changes the torus that acts on X, while α changes the torsor of the open orbit in X.

Let $M := \mathrm{Hom}(\overline{T}, \mathbb{G}_{m,\bar{k}})$ be the Γ_k-lattice that is the Cartier dual of T. Then one can use toric geometry to construct a flasque resolution using the stardard exact sequence. Let X be a choice of neutral smooth projective toric T-variety over k.

The set of prime $\overline{T}$-divisors in $\overline{X}$ carries an action of Γ_k, which factors through $\mathrm{Aut}(\Sigma)$. We denote this Γ_k-lattice by $\mathrm{Div}_{\overline{T}}(\overline{X})$. Then we have a flasque resolution of M:

$$0 \to M \to \mathrm{Div}_{\overline{T}}(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to 0,$$

For more details, see Voskresenskiĭ 1998, Section 6.

Lemma 4.4.1. Assume $\mathrm{D^b}(X)$ possesses a full exceptional collection. Then the Γ_k-module $\mathrm{K}_0(\overline{X})$ is permutation. If the collection is k-exceptional, then $\mathrm{K}_0(\overline{X})$ carries a trivial Γ-action.

Proof. If $E_1, \ldots, E_n$ is an exceptional collection, then over $\overline{k}$ we have

$$(E_i)_{\overline{k}} = \bigoplus (E_i^j)^{\oplus r_i},$$

where the E_i^j are distinct $\overline{k}$-exceptional objects permuted by Γ_k. The classes $[E_i^j]$ form a Γ-stable basis for $\mathrm{K}_0(\overline{X})$. There is no splitting and no action if all the exceptional objects are actually k-exceptional. □

We are now ready to prove the second main result.

Proof of Theorem 5. Since we have a surjective map $\det : \mathrm{K}_0(\overline{X}) \to \mathrm{Pic}(\overline{X})$ with $\mathrm{K}_0(\overline{X})$ carrying a trivial Γ-action, the module $\mathrm{Pic}(\overline{X})$ has trivial Γ-action. We always have a short exact sequence of Γ-modules

$$0 \to \hat{T} \to \mathrm{Div}(\bar{X}) \to \mathrm{Pic}(\bar{X}) \to 0.$$

Thus, taking Cartier duals and applying Theorem 4.2.2, we conclude that T is rational. $\square$

4.4.2 Producing linearizable étale collections

In this section, we identify a particular class of exceptional objects on split toric varieties which provide linearizable (up to sums) étale collections on any neutral form. We then give examples of non-retract rational smooth projective neutral toric varieties that possess full exceptional collections of this type.

Let $X(\Sigma)$ be a split smooth projective toric variety associated to a fan Σ. Let R denote the Cox ring of $X(\Sigma)$, so that

$$R \cong k[x_\rho \mid \rho \in \Sigma(1)].$$

The Cox ring is graded by $\mathrm{Pic}(X(\Sigma))$, where the weight of x_ρ is $\mathcal{O}(D_\rho) \in \mathrm{Pic}(X(\Sigma))$. We will identify weights with elements of $\mathrm{Pic}(X(\Sigma))$.

The finite group $\mathrm{Aut}(\Sigma)$ acts via homogeneous automorphisms on R. For a weight χ and graded R-module M, we let $M(\chi)$ be the graded R-module with $M(\chi)_\psi = M_{\chi+\psi}$.

Recall that $X(\Sigma) \cong U/\mathscr{D}(\mathrm{Pic}(X(\Sigma)))$ for a quasi-affine open subset U of $\mathrm{Spec}\, R$. As such, we have a restriction functor

$$j^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{Pic}}(\mathbb{A}^{\Sigma(1)}) \to \mathrm{D}^{\mathrm{b}}(X).$$

Definition 4.4.3. We say $X(\Sigma)$ has an exceptional collection of *TCI-type* if there exists a set of graded R-modules $F_1, \ldots, F_t$ such that

- for each $1 \leq s \leq t$
$$F_s = R(\chi_s)/(x_l \mid l \in I_s)$$
for some $\chi_s \in \operatorname{Pic}(X(\Sigma))$ and $I_s \subseteq \Sigma(1)$,
- the set $F_1, \ldots, F_t$ is $\operatorname{Aut}(\Sigma)$-stable, and
- the set of isomorphism classes of $j^*F_1, \ldots, j^*F_t$ forms a k-exceptional collection of $\mathrm{D}^{\mathrm{b}}(X(\Sigma))$.

Proposition 4.4.2. Let $X(\Sigma)$ be a split smooth projective toric variety over k with fan Σ and X a neutral smooth projective toric T-variety such that $X(\Sigma)_L \cong X_L$ for some extension L/k. If $X(\Sigma)$ posseses a full exceptional collection of TCI-type, then X possesses a full étale exceptional collection such each object is T-linearizable up to taking sums.

Proof. For the first statement, we need to check that the F_i descend to $\operatorname{Spec} R^{\operatorname{Gal}(L/k)}$. The action of $\operatorname{Aut}(\Sigma)$ on $\operatorname{Spec} R$ strictly permutes (i.e., up to equality and not isomorphism) modules of the form $R(\chi)/(x_1, \ldots, x_t)$. Thus, when we want to move to X by twisting via a cocycle from $H^1(k, \operatorname{Aut}(\Sigma))$, we have descent data for the orbits as Γ_k-equivariant modules. These descend to the exceptional objects whose endomorphisms are the étale algebras corresponding to the orbits as Γ_k-sets.

Now if we want to T-linearize the resulting objects we need to T_L-linearize the F_i in an $\operatorname{Aut}(\Sigma)$-stable manner. Clearly $R(\chi)/(x_1, \ldots, x_t)$ admits a T_L-linearization by lifting χ to a character of $\operatorname{Div}(X(\Sigma))$. Taking the orbit under the $\operatorname{Aut}(\Sigma)$-action gives a T-linearized object. Forgetting the linearization, we have just taken direct sums of the F_i. $\square$

Remark 4.4.2. Proposition 4.4.2 makes clear the difference between a full étale-exceptional collection of TCI-type on X and a full $\mathrm{Aut}(\Sigma)$-stable exceptional collection consisting of restrictions of line bundles to intersections of toric divisors.

Given an object of the form $L|_{D_1 \cap \cdots D_t}$ on $\overline{X}$, we can lift it to $R(\chi)/(x_1, \ldots, x_t)$. Let $H \leq \mathrm{Aut}(\Sigma)$ be the stabilizer of the subset $\{1, \ldots, t\} \subset \Sigma(1)$. Then for each $h \in H$, $h \cdot \chi = \chi + \chi_h$. This gives a class $(\chi_h) \in H^1(H, \mathrm{Ker}\, i^*)$, where $i^* : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(D_1 \cap \cdots \cap D_t)$ is the restriction map. We can promote an $\mathrm{Aut}(\Sigma)$-stable collection on $\overline{X}$ to an étale-exceptional collection of TCI-type on X if and only if $(\chi_h) = 0$ for all objects.

Next, we turn to identifying a particular split toric variety with a full exceptional collection of TCI-type and non-retract rational neutral forms. The split toric variety itself is very simple. It is the toric variety associated to the fan of Weyl chambers of the root system A_3 with its symmetry group $S_4 \times C_2$. This is also the decomposition group for the associated torus.

In Castravet and Tevelev 2017, Castravet and Tevelev construct full $\mathrm{Aut}(A_n)$-stable exceptional collections of sheaves for each of the split toric varieties corresponding to the root systems of type A, denoted $X(A_n)$. We recall those now.

An important idea in the construction are the cuspidal pieces of the derived categories of the $X(A_n)$. An object F of $\mathrm{D^b}(X(A_n))$ is called *cuspidal* if for all sub-root systems $A_\ell \leq A_n$ of type A, we have

$$\mathbf{R}\pi_* F = 0$$

where $\pi : X(A_n) \to X(A_\ell)$ is the corresponding map of toric varieties.

The collections constructed in Castravet and Tevelev 2017 are built inductively by pulling back the cuspidal pieces from subsystems of type A_ℓ for $\ell < n$ and then adding in the cuspidal part for n. We recall the collections in low dimensions.

- $X(A_0) = \mathrm{Spec}\, k$. The collection and whole cuspidal piece is $\mathcal{O}$.

- $X(A_1) = \mathbb{P}^1$. The collection in Castravet and Tevelev 2017 is $\{\mathcal{O}(-1), \mathcal{O}\}$ and the cuspidal piece is $\mathcal{O}(-1)$.

- $X(A_2)$ is del Pezzo surface of degree 6. Viewing $X(A_2)$ as the blowup of $\mathbb{P}^2$ at 3 non-colinear points, let H be the pullback of the hyperplane divisor on $\mathbb{P}^2$ and E_i the exceptional divisors, $i = 1, 2, 3$. Then the collection is given by

$$\{\mathcal{O}(-H), \mathcal{O}(-2H + E_1 + E_2 + E_3), \mathcal{O}(-H + E_1),$$

$$\mathcal{O}(-H + E_2), \mathcal{O}(-H + E_3), \mathcal{O}\}.$$

  The cuspidal part is $\mathcal{O}(-H), \mathcal{O}(-2H+E_1+E_2+E_3)$. The line bundles $\mathcal{O}(-H+E_1), \mathcal{O}(-H+E_2), \mathcal{O}(-H+E_3)$ are $\mathcal{O}(-1)$ pulled back from the three copies of A_1 in A_2, and of course $\mathcal{O}$ is pulled back from A_0.

- For $X(A_3)$, the collection consists of 1 line bundle pulled back from $X(A_0)$, 6 lines bundles coming from pulling back $\mathcal{O}(-1)$ from the six copies of A_1 in A_3, and $4 \times 2 = 8$ line bundles coming from pulling back $\mathcal{O}(-H), \mathcal{O}(-2H + E_1 + E_2 + E_3)$ from the four copies of A_2 in A_3, together with the cuspidal part.

  The cuspidal part breaks up into a block of 3 line bundles and 6 torsion sheaves. The line bundles are pulled back from the embedding of $X(A_3)$ into the wonderful compactification of the adjoint form of A_3 as the closure of the maximal torus.

  The torsion pieces can be described as follows. The divisors of $X(A_3)$ are the weights of A_3. The orbits are in bijection with nodes in the Dynkin diagram. There are six divisors corresponding to the middle node. Each such divisor, as a toric variety, is isomorphic to $X(A_1 \times A_1) \cong X(A_1) \times X(A_1)$. The torsion block consists of the $i_*\mathcal{O}(-1, -1)$ for each middle weight.

Lemma 4.4.2. Castravet and Tevelev's exceptional collection is of TCI-type. Therefore, we have a full étale-exceptional collection on any neutral form of $X(A_3)$ where all objects are T-linearizable up to passing to finite sums.

Proof. Everything except the torsion block is a line bundle, so we just need to check that this block lifts to $\operatorname{Spec} R$, with R the Cox ring, in an $\operatorname{Aut}(\Sigma)$-stable fashion.

A weight is in particular a linear function $\omega_D : \mathbb{Z}A_3 \to \mathbb{Z}$. The set of roots lying in the kernel of ω_D is a root system of type $A_1 \times A_1$. Hence, we have a map $\pi : X(A_3) \to X(A_1 \times A_1)$. The composition $\pi \circ i : X(A_1 \times A_1) \to X(A_1 \times A_1)$ is the identity Batyrev and Blume 2011, Remark 1.12.

The line bundle $\pi^*\mathcal{O}(-1,-1)$ therefore restricts via i^* to $\mathcal{O}(-1,-1)$. A computation identifies

$$\pi^*\mathcal{O}(-1,-1) \cong G_2^\vee(D + D')$$

where G_2 (using the notation of Castravet and Tevelev 2017) is $(S_4 \times C_2)$-fixed and D' is the image of D under the nontrivial element of C_2.

Let χ_{G_2}, χ, χ' be characters of $\mathscr{D}(\operatorname{Pic}(X(A_3)))$ corresponding to $G_2, \mathcal{O}(D), \mathcal{O}(D')$. Then, we can lift $i_*\mathcal{O}(-1,-1)$ to

$$R(-G_2 + D + D')/(x_D).$$

The action of $S_4 \times C_2$ permutes these choices of lifts. $\square$

Remark 4.4.3. We record some interesting observations about this collection that are not essential for the paper:

- the collections for A_n are also of TCI-type,
- the collection for A_3 is not strong, even if we shift the torsion sheaves by [1], and
- the collection does not form a window in the $\mathscr{D}(\mathrm{Pic})$-equivariant derived category of the spectrum of the Cox ring.

In general, can one find a collection of TCI-type forming a window?

We can restrict the decomposition group from $S_4 \times C_2$ to any subgroup $G \leq S_4 \times C_2$ and obtain full étale exceptional collections of the associated neutral form. In particular, we can restrict to $C_2 \times C_2$ to get such a collection on toric models for the associated norm one tori.

Lemma 4.4.3. Let L/k be a biquadratic extension. There exists a neutral smooth projective model X for $R^{(1)}_{L/k}\mathbb{G}_m$ such that $\mathrm{D}^{\mathrm{b}}(X)$ possesses a full étale exceptional collection where each object is $R^{(1)}_{L/k}\mathbb{G}_m$-linearizable up to a finite sum.

Proof. We can take $X(A_3)$ and restrict the decomposition group from $S_4 \times C_2$ to $C_2 \times C_2$. The exceptional collection given by Castravet and Tevelev is then a full $(C_2 \times C_2)$-stable exceptional collection of TCI-type by Lemma 4.4.2. Using Proposition 4.4.2, we get an étale collection on X. □

4.4.3 Main results

We are now ready to state and prove the main results relating rationality to full exceptional collections.

Theorem 4.4.1. Let X be a smooth projective T-toric variety over a field k with $X(k) \neq \emptyset$. Assume that

- X is not retract rational and
- the split form of X possesses a full k-exceptional collection of TCI-type.

Then there exists a field extension K/k and a T_K-torsor U such that ${}^{U}X_K$ has no K-points but has a full étale exceptional collection.

Proof. We can apply Proposition 4.2.2 to find an extension K/k such that $\text{Ж}(K, T_K)$ is nonempty. If $X(\Sigma)$ denotes the associated split form of X, then $X(\Sigma)_K$ admits a full K-exceptional collection of TCI-type. Hence, X_K has a full étale exceptional collection which is T-linearizable up to sums by Proposition 4.4.2. Twisting by each object in this collection gives a normalized cohomological invariant with target a Brauer group by Proposition 4.3.3. By definition, if we twist our exceptional collection by any element of $\text{Ж}(K, T_K)$, we still have an étale exceptional collection.

The twist ${}^{U}X$ has no rational points by Proposition 4.4.1. □

We can also be specific.

Corollary 4.4.1. Fix L/k with $\mathrm{Gal}(L/k) = S_4 \times C_2$ and let X be the associated neutral form of $X(A_3)$ over k and let T be the torus over k. There exists an extension K/k and a T_K-torsor U such that ${}^{U}X_K$ has no K-points but has a full étale exceptional collection.

Proof. This is an immediate application of Theorem 4.4.1 since X is not retract rational Kunyavskiĭ 1987 and possesses a full étale exceptional collection linearizable up to sums by Lemma 4.4.2. □

Proof of Theorem 4. From Proposition 4.2.2 and Proposition 4.2.3, we know that we can find infinitely-many biquadratic extension $L/\mathbb{Q}$ such that

$$\text{Ж}(k, R^{(1)}_{L/\mathbb{Q}}\mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}.$$

By Lemma 4.4.3, we have a neutral model for $R^{(1)}_{L/\mathbb{Q}}\mathbb{G}_m$ with a full exceptional collection of TCI-type. Appealing to Proposition 4.3.3 gives a full étale-exceptional collection on the nontrivial twist Y. The variety Y has no $\mathbb{Q}$-points by Proposition 4.4.1.

This covers the case of $d = 3$. For $d > 3$, note that if (X, Y) satisfy the conditions above then so does $(X \times Z, Y \times Z)$ for any rational Z. Taking such products finishes the proof. $\square$

Remark 4.4.4. Theorem 4 answers Bolognesi and Bernardara's Question 4.2.1 in the negative. Categorical representability in codimension 2 does not imply rationality for threefolds with negative Kodaira dimension. It does not even imply the existence of a point over the ground field. In particular, it also does not imply unirationality.

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